XIV. *On the Singular Solutions of Simultaneous Ordinary Differential Equations and the Theory of Congruencies.*

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INTRODUCTION.

§ 1. THIS paper is an attempt to show how the singular solutions of simultaneous ordinary differential equations are to be found either from a complete primitive or from the differential equations.

The *latter* question has been treated by MAYER ('Math. Ann.,' vol. 22, p. 368) in a somewhat different way, but with the same result. He also gives a reference to a paper in Polish by ZAJAÇZKOWSKI (summarized in vol. 9 of the 'Jahrbuch der Fortschritte der Mathematik), and to one by SERRET in vol. 18 of 'LIOUVILLE'S Journal.'

The general result is that there may be as many forms of solution as there are variables (the differential equations being of the first order, to which they may always be reduced). Each form is derived from the one before by the process of finding the envelope, and each contains fewer arbitrary constants by one than the form from which it is directly derived.

The general theory is given in §§ 2, 3, and in § 4 it is shown how the singular solutions are to be formed from the differential equations themselves. In §§ 5–9 the theory is connected with that of consecutive solutions belonging to the complete primitive. §§ 10–13 are taken up with geometrical interpretations relating to plane curves and also to curves in space of $n + 1$ dimensions, $n + 1$ being the number of variables. In §§ 14–16 the case is discussed in which a system of singular solutions is included in a former system or in the complete primitive.

The rest of the paper contains the application of the theory to certain examples. The first example (§§ 17–21) is the case of the lines in two osculating planes of a twisted curve, and in particular of a twisted cubic. The particular example is given by MAYER and SERRET. The second (§§ 22–26) is that of the congruency of common tangents to two quadric surfaces, and generally (§§ 27–38) of the bitangents to any surface. The third (§§ 39–50) is that of the essentially different kind of congruency which consists of the inflexional tangents to a surface. It seems natural to call these two kinds of congruency *bitangential* and *inflexional* respectively. The fourth example (§§ 51, 52) is that of a system of conics touching six planes. The fifth (§§ 53–56) is that of a doubly infinite system of parabolas in one plane, the differential equation being a case of an extension of CLAIRAUT's form $y = px + f(p)$, which is explained in §§ 53–55.

General Theory. (§§ 2, 3.)

§ 2. Suppose that we have n ordinary simultaneous differential equations, involving one independent variable x, and n dependent variables $y_1, y_2, \ldots y_n$, with their first differential coefficients $p_1, p_2, \ldots p_n$.

The "complete" solution of such a system will consist of n equations involving $x, y_1, y_2, \ldots y_n$, and n arbitrary constants, $c_1, c_2, \ldots c_n$.

Suppose that such a solution is known. The question then arises, "Are there any other solutions which it does not include?" This is the question that we now seek to answer.

If we take the differential equations in the form

$$f_r(x, y_1, y_2, \ldots y_n, p_1, \ldots p_n) = 0 \quad (r = 1, 2 \ldots n) \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

and the integrals as

$$\mathrm{F}_r(x, y_1, \ldots y_n, c_1, \ldots c_n) = 0 \quad (r = 1, 2 \ldots n) \quad . \quad . \quad . \quad . \quad \text{(II.)},$$

we have by differentiating

$$\frac{\partial \mathrm{F}_r}{\partial x} + p_1 \frac{\partial \mathrm{F}_r}{\partial y_1} + \ldots + p_n \frac{\partial \mathrm{F}_r}{\partial y_n} = 0 \quad (r = 1, 2 \ldots n \quad . \quad . \quad . \quad \text{(III.)}.$$

Let the system

$$\lambda_r(x, y_1, \ldots y_n) = 0 \quad (r = 1, 2 \ldots n) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

be an integral of (I.).

From (IV.) we derive

$$\frac{\partial \lambda_r}{\partial x} + p_1 \frac{\partial \lambda_r}{\partial y_1} + \ldots + p_n \frac{\partial \lambda_r}{\partial y_n} = 0 \quad (r = 1, 2 \ldots n) \quad . \quad . \quad . \quad . \quad \text{(V.)}.$$

By eliminating $p_1, \ldots p_n$ from (III.) and (V.) we find such values of $c_1, c_2, \ldots c_n$ as will make (III.) and (V.) equivalent*; but if the values of $p_1, \ldots p_n$ given by (III.)

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* This argument must be somewhat modified if the equations (III.) are not enough to give the values of $c_1, c_2, \ldots c_n$. Suppose that m independent equations, and no more, can be formed from (III.) not containing the quantities $c_1, \ldots c_n$. These equations must be included in the system (I.), and are therefore equally satisfied (possibly in virtue of (IV.)) by the values of $p_1, \ldots p_n$ given by (V.).

The other $n - m$ equations of the system (III.) may then be transformed by substitution of the values and (V.) respectively are substituted in (I.), the results are respectively equivalent to (II.) and (IV.). Hence (II.) and (IV.) are also made equivalent, and therefore (IV.) may be derived from (II.) by substituting appropriate functions of x for $c_1, \dots c_n$.

Hence, by subtracting (III.) from the derivative of (II.), we have

$$\sum_{s=1}^{n} \frac{\partial F_r}{\partial c_s} \frac{dc_s}{dx} = 0 \quad (r = 1, 2 \dots n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VI.}).$$

Thus, unless $c_1, c_2 \dots c_n$ are all constants,

$$\frac{\partial (F_1, F_2 \dots F_n)}{\partial (c_1, c_2 \dots c_n)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VII.}).$$

§ 3. The equations (II.) and (VII.) may be considered as defining $x, y_1, \dots y_n$ in terms of $c_1, c_2 \dots c_n$, and the equations (VI.) will then give the ratios $dc_1 : dc_2 : \dots : dc_n$ in terms of $c_1, c_2 \dots c_n$. We thus have $n - 1$ differential equations connecting the n quantities $c_1, \dots c_n$. Their complete primitive will involve $n - 1$ arbitrary constants, and by eliminating $c_1 \dots c_n$ from (II.) and (VII.) and this complete primitive we have a solution of (I.) involving $n - 1$ arbitrary constants, which we may call the *first singular solution* of (I.)

The differential equations given by (VI.) may have a first singular solution involving $n - 2$ arbitrary constants, and from this we should derive the *second singular solution* of (I.) by eliminating $c_1 \dots c_n$ as before.

This process may go on till we have n singular solutions, with $n - 1, n - 2 \dots 2, 1, 0$ arbitrary constants respectively, or it may stop at any stage. If, for instance, the left-hand side of the equation (VII.) were an absolute constant, there would be no singular solution at all.

Formation of Singular Solutions from the Differential Equations.

§ 4. The equation (VII.) is the condition that two solutions of (II.) when solved for $c_1, c_2 \dots c_n$ shall coincide. But, generally, when this happens, the equations (III.) give coincident sets of values for $p_1, p_2 \dots p_n$.

of $p_1 \dots p_n$ from (V.), and become relations (A) connecting $c_1, c_2, \dots c_n$ with x, and $y_1, y_2, \dots y_n$ which are given as functions of x by (IV.).

Now by substituting the values of $p_1 \dots p_n$ from (III.) in (I.), we have integral equations which must be included in (II.).

The number of these equations will be $n - m$, because m are already satisfied identically.

These equations are also satisfied if (A) and (IV.) are; for the values of $p_1 \dots p_n$ given by (III.) and (V.) are then the same and those given by (V.) satisfy (I.).

The system (II.) will supply exactly m further equations which in combination with (A) and (IV.) give the values of $c_1, c_2, \dots c_n$ in terms of x.

If not, that is if the two sets of values of $p_1, \dots$ are different, all the determinants of the matrix

$$\left| \begin{array}{llll} \frac{\partial F_1}{\partial x}, & \frac{\partial F_1}{\partial y_1}, & \dots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial x}, & \frac{\partial F_2}{\partial y_1}, & \dots & \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial F_n}{\partial x} & \dots & & \frac{\partial F_n}{\partial y_n} \end{array} \right|$$

must vanish (compare § 11, below.)

The equations (III.) are then not enough to determine $p_1, p_2 \dots p_n$, and the conditions (VI.) do not ensure that the system of equations found by the above process will satisfy (I.) and be a solution at all.

In general, then, for a singular solution, the two sets of values of $p_1, p_2 \dots p_n$ are coincident. But these are given by the equations (I.).

Hence generally the equation

$$\frac{\partial (f_1, f_2, \dots f_n)}{\partial (p_1, p_2, \dots p_n)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII.)*}$$

is satisfied by a singular solution.

We have, therefore, the following process for finding the singular solution from the differential equations (I.):—

Form the equation (VIII.) and let $E = 0$ be the result of eliminating $p_1, p_2, \dots p_n$ from (I.) and (VIII.). Suppose that ϕ is a factor of E such that the equation $d\phi/dx = 0$ can be deduced, by substitution without differentiating, from $\phi = 0$ and (I.). Then, by treating $\phi = 0$ as if it were a particular first integral† of (I.), which is now allowable, reduce (I.) to a system of $n - 1$ differential equations in n variables. The complete integral of this system belongs to the first singular solution of (I.). If ϕ is the only factor of E that satisfies the condition of being an integral, then it yields the whole of the first singular solution.

The first singular solution of the new system of $n - 1$ equations gives the second singular solution of the original system of n equations and so on (see also § 15, below).

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* MAYER finds this equation as the condition that the second and following differential coefficients, as given by the equations (I.), should be indeterminate, and thus shows that (VIII.) must *always* be satisfied by a singular solution.

† If the phrase "first integral" is restricted to such as involve only one arbitrary constant, we may use "singular first integral" for such an equation as $\phi = 0$ is here supposed to be.

Analytical Connection of Different Solutions. (§§ 5–9.)

§ 5. Let us write J for $\partial(f_1, f_2, \ldots f_n)/\partial(p_1, p_2, \ldots p_n)$ and $\lambda_1, \lambda_2, \ldots \lambda_n$ for the minors of $\partial f_n/\partial p_1, \partial f_n/\partial p_2, \ldots \partial f_n/\partial p_n$ in J.

The equations $f_1 = 0, f_2 = 0, \ldots f_n = 0$, $\mathrm{J} = 0$ give $p_1, p_2, \ldots p_n, y_n$ as functions of $x, y_1, \ldots y_{n-1}$, and, for the second singular solution, we have to suppose that two solutions coincide, that is, we put

$$\partial(f_1, f_2, \ldots f_n, \mathrm{J})/\partial(p_1, p_2, \ldots p_n, y_n) = 0.$$

Since $\mathrm{J} = 0$, this equation may be reduced, by multiplication by λ_n, to

$$\left(\lambda_1 \frac{\partial \mathrm{J}}{\partial p_1} + \lambda_2 \frac{\partial \mathrm{J}}{\partial p_2} + \ldots + \lambda_n \frac{\partial \mathrm{J}}{\partial p_n}\right) \frac{\partial(f_1, f_2, \ldots f_n)}{\partial(p_1, \ldots p_{n-1}, y_n)} = 0.$$

Now, y_n may equally well be replaced by $y_1, y_2 \ldots$, or y_{n-1}, so that the condition sought is given by the first factor, which we shall call J_1.

The equations $f_1 = 0, \ldots f_n = 0$, $\mathrm{J} = 0$, $\mathrm{J}_1 = 0$ give $p_1, p_2, \ldots p_n, y_{n-1}, y_n$ as functions of $x, y_1, \ldots y_{n-2}$, and, if the values of p_{n-1}, p_n, given by differentiating those of y_{n-1}, y_n, agree with the values given by the solution of the equations, we are to find the second singular solution by integrating the equations that give $p_1, \ldots p_{n-2}$.

To find the third singular solution, we have to make the system $f_1 = 0, \ldots f_n = 0$, $\mathrm{J} = 0$, $\mathrm{J}_1 = 0$ have equal roots. The condition for this is found in the same way* to be

$$\lambda_1 \frac{\partial \mathrm{J}_1}{\partial p_1} + \lambda_2 \frac{\partial \mathrm{J}_1}{\partial p_2} + \ldots + \lambda_n \frac{\partial \mathrm{J}_1}{\partial p_n} = 0,$$

and so we may pass on to the other singular solutions, if any.

§ 6. As $\mathrm{J} = \lambda_1 \frac{\partial f_n}{\partial p_1} + \lambda_2 \frac{\partial f_n}{\partial p_2} + \ldots + \lambda_n \frac{\partial f_n}{\partial p_n}$, we may write for the integrals that have to be taken to give the r^{th} singular solution

$$\left(\Sigma\lambda \frac{\partial}{\partial p}\right)^s f_n = 0 \quad (s = 1, 2, \ldots r).$$

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* The condition is

$$\frac{\partial(f_1, f_2 \ldots f_n, \mathrm{J}, \mathrm{J}_1)}{\partial(p_1, p_2 \ldots p_n, y_n, y_{n-1})} = 0.$$

Multiplied by λ_n this becomes

$$\Sigma\lambda_r \frac{\partial \mathrm{J}_1}{\partial p_r} \times \frac{\partial(f_1 \ldots f_{n-1}, \mathrm{J}, \mathrm{J}_1)}{\partial(p_1 \ldots p_{n-1}, y_n, y_{n-1})} = 0.$$

The second factor, being unsymmetrical, is irrelevant.

We have also identically

$$\left(\Sigma\lambda\frac{\partial}{\partial p}\right)f_i = 0 \quad (i = 1, 2, \ldots n - 1),$$

and therefore

$$\left(\Sigma\lambda\frac{\partial}{\partial p}\right)^s f_i = 0.$$

Thus

$$\left(\Sigma\lambda\frac{\partial}{\partial p}\right)^s (\mathrm{A}_1 f_1 + \mathrm{A}_2 f_2 + \ldots + \mathrm{A}_n f_n) = 0,$$

where $\mathrm{A}_1, \mathrm{A}_2 \ldots \mathrm{A}_n$ are arbitrary functions.

These may be so chosen that $\Sigma \mathrm{A}f$ is the eliminant of $f_1, f_2 \ldots f_n$ with respect to $p_2, \ldots p_n$. We will call this eliminant P_1; it does not involve $p_2 \ldots p_n$.

The equations then become

$$\left(\lambda_1\frac{\partial}{\partial p_1}\right)^s \mathrm{P}_1 = 0 \quad (s = 1, 2, \ldots r),$$

that is to say

$$\left(\frac{\partial}{\partial p_1}\right)^s \mathrm{P}_1 = 0 \quad (s = 1, 2, \ldots r).$$

Thus the equation $\mathrm{P}_1 = 0$ gives $r + 1$ coincident values of p_1. The same holds for $p_2, p_3, \ldots p_n$.

If we take a system of values of $x, y_1, \ldots y_n$ such that s consecutive members* of the r^{th} singular system of solutions are satisfied, then $s - 1$ consecutive members of the $(r + 1)^{\text{th}}$ will be satisfied, $s - 2$ of the $(r + 2)^{\text{th}}$ and so on to the $(r + s - 1)^{\text{th}}$, after which none are satisfied. Also $s + 1$ of the $(r - 1)^{\text{th}}$ system will be satisfied, $s + 2$ of the $(r - 2)^{\text{th}}$ and so on, and lastly $s + r$ of the complete primitive system.

§ 7. The second singular solution of (I.) is the first of (VI.) and (VII.) Now by (VI.) the ratios $dc_1 : dc_2 : \ldots : dc_n$ are given rationally in terms of $c_1, c_2 \ldots c_n$, and certain other variables $x, y_1 \ldots y_n$ which are connected with $c_1, \ldots c_n$ by the equations (II.) and (VII.) The condition that (II.) and (VII.) shall have two consecutive solutions is

$$\frac{\partial(\mathrm{F}_1, \ldots \mathrm{F}_n, \Omega)}{\partial(x, y_1, \ldots y_n)} = 0,$$

Ω being written for the left-hand side of (VII.).

This equation we may write

$$\left(\frac{\partial}{\partial x} + p_1\frac{\partial}{\partial y_1} + p_2\frac{\partial}{\partial y_2} + \ldots\right)\Omega = 0 \text{ or } \Delta\Omega = 0,$$

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* Different members of the system are got by giving different sets of values to the arbitrary constants.

where $p_1, p_2, \ldots p_n$ have the values given by the equations (III.), and $c_1, c_2 \ldots$ are treated as constants in the differentiation.

Since $\Omega = 0$, we may use instead of the operator Δ, another, ∇, such that

$$\nabla = \frac{dc_1}{dx}\frac{\partial}{\partial c_1} + \frac{dc_2}{dx}\frac{\partial}{\partial c_2} + \ldots$$

for $\Delta\Omega + \nabla\Omega = 0$.

By comparing $\nabla\Omega = 0$ with (VI.), we see that all the determinants of the matrix

$$\left|\begin{array}{llll} \dfrac{\partial\Omega}{\partial c_1}, & \dfrac{\partial F_1}{\partial c_1}, & \dfrac{\partial F_2}{\partial c_1}, \ldots & \dfrac{\partial F_n}{\partial c_1} \\ \dfrac{\partial\Omega}{\partial c_2}, & \dfrac{\partial F_1}{\partial c_2}, \ldots & & \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial\Omega}{\partial c_n}, & \dfrac{\partial F_1}{\partial c_n}, \ldots & & \end{array}\right|$$

vanish.

§ 8. For the third singular solution we take the further integral equation $\Delta^2\Omega = 0$ or $\nabla^2\Omega = 0$, which forms are equivalent, since already $\Omega = 0$, $\Delta\Omega = 0$. The argument is the same as in § 6, and may be carried on till the last singular solution is reached.

The equations $\Omega = 0 = \nabla\Omega = \nabla^2\Omega = \ldots = \nabla^{r-1}\Omega$, which yield the r^{th} singular solution, show that the equations $F_1 = 0 = F_2 = F_3 = \ldots = F_n$, when solved for $c_1, c_2, \ldots c_n$, have $r + 1$ coincident solutions.

§ 9. The equations in the other form, viz., $\Omega = 0$, $\Delta\Omega = 0 \ldots \Delta^{r-1}\Omega = 0$, show that the system

$$F_1 = 0 = F_2 = \ldots = F_n = \Omega$$

is satisfied by $r + 1$ consecutive sets of values of $x, y_1, \ldots y_n$.

For suppose E to be the eliminant of $F_1, F_2 \ldots \Omega$ with respect to $c_1, \ldots c_n$, we have then an identity

$$E = A_1F_1 + A_2F_2 + \ldots + A_nF_n + B\Omega.$$

Differentiate partially as to $c_1, c_2 \ldots c_n$ in turn, and we have

$$B\frac{\partial\Omega}{\partial c_s} + \sum_{r=1}^{n} A_r\frac{\partial F_r}{\partial c_s} = -\Omega\frac{\partial B}{\partial c_s} - \sum_{r=1}^{n} F_r\frac{\partial A_r}{\partial c_s} \quad (s = 1, 2 \ldots n)$$

Eliminating $A_2, \ldots A_n$ on the left-hand side, we find

$$B\frac{\partial(\Omega, F_2, \ldots F_n)}{\partial(c_1, \ldots c_n)} + A_1\Omega = -\Omega\frac{\partial(B, F_2, \ldots F_n)}{\partial(c_1, \ldots c_n)} - \sum_{r=1}^{n} F_r\frac{\partial(A_r, F_2, \ldots F_n)}{\partial(c_1, \ldots c_n)}.$$

Now, in general, the coefficient of B on the left does not vanish when $\Omega = 0 = F_1 = \ldots = F_n$, and therefore B must vanish for such values, and must be of the form

$$C_1F_1 + C_2F_2 + \ldots + C\Omega.$$

Thus with a slight change in the meaning of $A_1, A_2 \ldots$,

$$E = C\Omega^2 + A_1F_1 + \ldots + A_nF_n.$$

Now, $\Delta F_1 = 0 = \Delta F_2 = \ldots$ identically, and $\Omega = 0, \Delta\Omega = 0 \ldots \Delta^{r-1}\Omega = 0$ for systems of values that satisfy one of the r^{th} system of singular solutions. Hence in such a case

$$E = 0, \Delta E = 0 \ldots \Delta^r E = 0,$$

and the number of consecutive solutions of the equations

$$E = 0 = F_1 = F_2 = \ldots = F_n \text{ is } r + 1.$$

Geometrical Interpretations. (§§ 10–13.)

§ 10. The geometrical application of the above theory to curves in space of $n + 1$ dimensions is easy.

The equations (II.) may be taken to represent a series of curves (that is, singly infinite continuous series of points) in such space. Through any point a certain number, i, of such curves may be drawn, i being the number of solutions of (II.) when solved for $c_1, c_2, \ldots c_n$.

At every point such a curve is met by $i - 1$ other curves of the system, and at certain points one of these $i - 1$ curves coincides with it.

The direction of the tangent to such a curve is given by the equation (I.) or (III.).

The first singular solution is the envelope of a series of curves of the system, each of which meets the consecutive one, and the locus of all such envelopes is the surface (n^{ply} infinite series of points) whose equation is $E = 0$ (see § 4).

In general, the r^{th} singular solution is the envelope of a series of curves belonging to the $(r - 1)^{\text{th}}$ singular system, and such that each meets the consecutive one; the locus of such envelopes is an $(n - r + 1)^{\text{ply}}$ infinite continuous series of points at each of which $r + 1$ consecutive curves of the original system meet, as also do $r - s + 1$ of the s^{th} singular system. All the successive singular curves touch the original curve.

§ 11. If every curve of the first system (II.) has a node, the equation (VII.) will be satisfied at every point of the node locus, but (VIII.) will not (compare § 4). If every curve has a cusp (VII.) and (VIII.) will be satisfied by the equation to the cusp locus, but it will not fulfil the condition of being a first integral. If different curves of the system touch, the tac-locus will satisfy the equation (VIII.) but not (VII.). All this is exactly parallel to the known theory of single differential equations of the first order.

§ 12. There is also an application to plane curves.

Suppose that the differential equations (I.) include the following:—

$$p_1 = y_2, p_2 = y_3 \dots p_{n-1} = y_n.$$

Then the system (I.) simply reduces to an equation of the n^{th} order and its n first integrals make up the system (II.) from which the singular solutions are derived.

If we write the differential equation

$$f(x, y, p_1, p_2 \dots p_n) = 0,$$

putting y for y_1, then the equation (VIII.) becomes

$$\partial f / \partial p_n = 0,$$

and this is the first integral (if it is one) from which the singular solutions are derived.

The final integral, found by eliminating $p_1, p_2 \dots p_{n-1}$ from the system (II.), is the equation to a system of curves, of which there are i passing through any point and having at that point contact of the $(n-1)^{\text{th}}$ order with any assigned curve through it; all these have contact with one another of the $(n-1)^{\text{th}}$ order.

If two of them coincide, then a curve of the first singular system passes through the point and has contact of the $(n-1)^{\text{th}}$ order with each of the i curves, and of the n^{th} order with either of the two coincident ones.

A curve of the first singular system can be made to have contact of the $(n-2)^{\text{th}}$ order with any given curve at any point of it.

At any point of a curve of the second singular system three coincident curves of the original system and two of the first singular system will satisfy the conditions for contact with it of the orders $n-1$, $n-2$, respectively, and in each case the contact will be actually of the n^{th} order, and so in general. The single curve of the n^{th} singular system is the envelope of those of the $(n-1)^{\text{th}}$, but it is more than an ordinary envelope since its contact with each of the enveloped curves is of the n^{th} order.

§ 13. It should be noticed that if $n-1$ of the dependent variables are eliminated by differentiation from n simultaneous equations, the new equation, of order n, will be satisfied by the same complete primitive, but that its singular solutions will generally be different, since the singular solutions of the system do not furnish the same values as the complete primitive for any but the first differential coefficients (compare § 17 below).

Singular Solutions Included in the Complete Primitive. (§§ 14–16.)

§ 14. In certain cases it appears to be possible for the second singular solution to exist without the first.

The ratios $dc_1 : dc_2 \ldots$ are given by the equations (VI.) and involve $x, y_1, y_2 \ldots y_n$, which are given in terms of $c_1, c_2 \ldots$ by the equations (II.) and (VII.). If these are not enough, that is to say, if Ω can be expressed in terms of $c_1, c_2 \ldots c_n$, there is no first singular solution.

For, from $\Omega = 0$, may be deduced

$$\sum_{r=1}^{n} \frac{\partial \Omega}{\partial c_r} dc_r = 0,$$

a further linear equation to be satisfied by $dc_1, dc_2, \ldots$ Thus either $c_1, c_2 \ldots$ are all constants, or the further integral equation

$$\frac{\partial (\mathrm{F}_1, \mathrm{F}_2 \ldots \mathrm{F}_{n-1}, \Omega)}{\partial (c_1, c_2 \ldots c_n)} = 0,$$

whose left-hand side we shall call Ω_1, is satisfied.

In virtue of this equation $\Omega = 0$ is an integral of the equations (VI.). The values of $x, y_1 \ldots y_n$ are given in terms of $c_1, \ldots c_n$ by (II.) and $\Omega_1 = 0$, and by substituting these values in (VI.) and finding $n - 2$ more integrals of the equations so derived, we get the second singular solution, containing $n - 2$ arbitrary constants.

It would, perhaps, be better to say that in such a case the first singular solution is included in the complete primitive, the values of the arbitrary constants being so chosen as to satisfy the equation $\Omega = 0$.

If Ω_1 can be expressed as a function of $c_1, \ldots c_n$ only by means of the equations (II.), we must use the equation

$$\frac{\partial (\mathrm{F}_1, \ldots \mathrm{F}_{n-1}, \Omega_1)}{\partial (c_1, \ldots c_n)} = 0, \text{ or } \Omega_2 = 0,$$

to find the third singular solution, the second being included in the first.

Because $\Omega_2 = 0$, $\Omega_1 = 0$ is an integral of (VI.), and because $\Omega_1 = 0$, $\Omega = 0$ is an integral. Therefore $n - 3$ integrals are still to be found.

This process may be carried on as far as is needed. It is also to be used if Ω has any factor that does not involve $x, y_1 \ldots y_n$.*

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* It sometimes happens that the equations (I.) and (VII.) are enough to define $x, y_1, \ldots y_n$ in terms

§ 15. The case of § 14 presents itself quite naturally if we start from the differential equations (I.) instead of from the integrals (II.). Let ϕ, as before, be a factor of E.

There is no apparent reason why the equation

$$\frac{d\phi}{dx} = 0$$

should follow algebraically from the system (I.) and the equation $\phi = 0$.

Let $E_1 = 0$ be the result of eliminating $p_1, p_2 \ldots p_n$ from the system (I.) and the equation $d\phi/dx = 0$.

Let ϕ_1 be a factor of E_1 and let $E_2 = 0$ be the result of eliminating $p_1, \ldots p_n$ from the system (I.) and $d\phi_1/dx = 0$, and so on for $\phi_2, \phi_3 \ldots$

There is no apparent reason why any function in the series $\phi, \phi_1, \phi_2 \ldots$ should vanish because all those before it are supposed to vanish. If one of them, say ϕ_r, does satisfy this condition, then the equations

$$\phi = 0 = \phi_1 = \phi_2 = \ldots = \phi_{r-1}$$

are integrals of (I.), and by using them and finding $n - r$ other integrals, each containing an arbitrary constant, we have a singular solution of the r^{th} system.

§ 16. Thus, as in the simpler case when there is a single equation of the first order, the existence of singular solutions appears to be the rule if we consider the integrals, the exception if we consider the differential equations, and in fact the number of conditions for the absence of the first r singular solutions rises with r from the first point of view, while the generality of the conditions for the existence of the $(r + 1)^{\text{th}}$ from the second point of view decreases as r increases.

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of $c_1, c_2 \ldots c_n$, but that when $x, y_1 \ldots y_n$ are eliminated from (VI.) by this means, an integral equation presents itself as an alternative to a differential equation. In such a case the integral equation will, of course, relate to the second singular solution, for it contains no arbitrary constant.

Considered geometrically, the second singular solution thus arising will be enveloped by the complete primitive, and, therefore, also by the first singular solution (for both lie on the locus of singular solutions $E = 0$), although it may not be a singular solution of the *differential* equations (VI.).

A proper change in the forms of the arbitrary constants would reduce this case to the ordinary one. If the integral factor is $\theta^{r-1}\,\phi^{s-1} \ldots$, $\theta, \phi \ldots$ being functions of $c_1, c_2 \ldots$ then if in the system of arbitrary constants we take $\theta^r\,\phi^s \ldots$ instead of c_1, the factor will disappear.

In fact the ordinary equation

$$y = px - p^2,$$

or any other, may be transformed so that its singular solution appears as an integral factor. Put

$$y = \tfrac{1}{4}(x^2 - z^2)$$

and we find

$$y - px + p^2 = \tfrac{1}{4}z^2(z'^2 - 1).$$

Examples. I. *Congruency of Bitangents to a Torse.* (§§ 17–19.)

§ 17. The simplest examples are equations of CLAIRAUT's form, such as

$$\left.\begin{aligned} y_1 &= p_1x + p_1^2 + p_1p_2 + p_2, \\ y_2 &= p_2x + p_1p_2 + p_2^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\alpha).$$

The complete primitive is

$$\left.\begin{aligned} y_1 &= c_1x + c_1^2 + c_1c_2 + c_2, \\ y_2 &= c_2x + c_1c_2 + c_2^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\beta).$$

The singular solutions are given by the equations

$$\left.\begin{aligned} 0 &= (x + 2c_1 + c_2)\, dc_1 + (c_1 + 1)\, dc_2, \\ 0 &= c_2\, dc_1 + (x + c_1 + 2c_2)\, dc_2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\gamma).$$

Hence, by eliminating x,

$$(c_2\, dc_1 - c_1\, dc_2)\,(dc_1 + dc_2) = dc_2^2 \quad \ldots\ldots\ldots \quad (\delta).$$

This is also of CLAIRAUT's form, and its integral is

$$(1 - \mu)\, c_2 - \mu c_1 = \mu^2 \quad \ldots\ldots\ldots \quad (\epsilon),$$

so that

$$dc_2 : dc_1 :: \mu : 1 - \mu.$$

Thus

$$\begin{aligned} &(x + 2c_1 + c_2)\,(1 - \mu) + \mu\,(c_1 + 1) = 0, \\ &2c_1 = -\mu^2 - \mu - x + \mu x, \\ &2c_2 = \mu^2 - \mu x, \end{aligned}$$

$$\left.\begin{aligned} 4y_1 &= (\mu - x)\,(\mu^2 + 3\mu + x - x\mu), \\ 4y_2 &= -\mu\,(\mu - x)^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\zeta).$$

The equations (ζ) furnish the first singular solution. For the second we must make (ϵ), as a quadratic in μ, have equal roots, that is, we must put

$$\begin{aligned} 2\mu &= -c_1 - c_2, \\ 2c_2 &= \mu\,(c_1 + c_2). \end{aligned}$$

From these and the former equations we find

$$\mu = \tfrac{1}{3}x, \qquad c_1 = \tfrac{1}{9}x^2 - \tfrac{2}{3}x, \qquad c_2 = -\tfrac{1}{9}x^2.$$

$$\left.\begin{aligned} y_1 &= \tfrac{1}{27}x^3 - \tfrac{1}{3}x^2, \\ y_2 &= -\tfrac{1}{27}x^3 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (\eta).$$

The equations (η) are the second singular solution.

We have here an example of what was pointed out in § 13. If we differentiate the equations (α) and eliminate y_2 and p_2, the resulting *differential equation of the second order* is

$$d^2y_1/dx^2 = 0.$$

This is satisfied by the complete primitive (β), but not by either of the singular solutions.

§ 18. To find the singular solutions from the original equations, we have first to try if the equation

$$\begin{vmatrix} x + 2p_1 + p_2, & p_1 + 1 \\ p_2, & x + 2p_2 + p_1 \end{vmatrix} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (\theta)$$

can be used as a first integral.

The elimination of p_1 and p_2 gives

$$4(y_1 + y_2)^3 + x^2(y_1 + y_2)^2 + 4x^3y_2 + 18xy_2(y_1 + y_2) - 27y_2^2 = 0 \; . \; . \quad (\iota).$$

It is easily verified that the derivative of this equation is an algebraical consequence of it in virtue of the original differential equations.

§ 19. If we take x, y_1, y_2 as Cartesian co-ordinates, the equations (β) represent a line common to two osculating planes of the twisted cubic (η), (ζ) represent the conic enveloped by this line when one of the osculating planes is fixed and the other variable, and (ι) is the equation to the torse enveloped by the osculating planes.

The General Case. (§§ 20, 21.)

§ 20. For any other twisted curve there is a similar theory.

Let

$$x + Ay_1 + By_2 + C = 0$$

be the equation to an osculating plane, A, B, C being functions of a parameter μ. Let A_r, B_r, C_r denote the same functions of μ_r, and let A'_r be written for $dA_r/d\mu_r$.

Then by eliminating μ_1 and μ_2 from the four equations

$$1 + A_1 p_1 + B_1 p_2 = 0, \qquad x + A_1 y_1 + B_1 y_2 + C_1 = 0,$$
$$1 + A_2 p_1 + B_2 p_2 = 0, \qquad x + A_2 y_1 + B_2 y_2 + C_2 = 0,$$

we have two relations which may be put in the Clairaut form

$$y_1 = p_1 x + \phi_1 (p_1, p_2),$$
$$y_2 = p_2 x + \phi_2 (p_1, p_2).$$

§ 21. The complete primitive of these equations will represent the trace of any one osculating plane on any other. The singular solutions are given by

$$(A'_1 y_1 + B'_1 y_2 + C'_1) \frac{d\mu_1}{dx} = 0, \qquad (A'_2 y_1 + B'_2 y_2 + C'_2) \frac{d\mu_2}{dx} = 0.$$

For the first singular solution, either μ_1 or μ_2 is to be equated to a constant, while the other satisfies the equation

$$A' y_1 + B' y_2 + C' = 0.$$

The first singular solution is, therefore, the trace on any osculating plane of the torse which they all envelope.

For the second we must suppose both μ_1 and μ_2 to satisfy the equation

$$A' y_1 + B' y_2 + C' = 0.$$

This gives the original curve, which is the cuspidal edge of the torse, together with the nodal curve of the torse. The latter is not a solution of the differential equations.

Example II. A Bitangential Congruency. (§§ 22-26.)

§ 22. Take for another example the equations

$$(A + Bp_1^2 + Cp_2^2) \{B (y_1 - p_1 x)^2 + C (y_2 - p_2 x)^2 - 1\} = \{Bp_1 (y_1 - p_1 x) + Cp_2 (y_2 - p_2 x)\}^2,$$
$$(a + bp_1^2 + cp_2^2) \{b (y_1 - p_1 x)^2 + c (y_2 - p_2 x)^2 - 1\} = \{bp_1 (y_1 - p_1 x) + cp_2 (y_2 - p_2 x)\}^2.$$

These define $y_1 - p_1 x$, and $y_2 - p_2 x$ as functions of p_1 and p_2, and are, therefore, of the Clairaut form, and may be integrated by substituting c_1 for p_1 and c_2 for p_2.

The integral in this form will represent four lines of a congruency, which consists in fact of all the common tangents to the two quadric surfaces

$$\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 = 1,$$
$$ax^2 + by_1^2 + cy_2^2 = 1.$$

The first of the differential equations may be written

$$(\mathrm{A} + \mathrm{B}p_1^2 + \mathrm{C}p_2^2)(\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1) = (\mathrm{A}x + \mathrm{B}y_1p_1 + \mathrm{C}y_2p_2)^2.$$

One of the integrals is, therefore,

$$(\mathrm{A} + \mathrm{B}c_1^2 + \mathrm{C}c_2^2)(\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1) = (\mathrm{A}x + \mathrm{B}y_1c_1 + \mathrm{C}y_2c_2)^2.$$

Thus, one of the equations that give the singular solutions is

$$(\mathrm{B}c_1\,dc_1 + \mathrm{C}c_2\,dc_2)(\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1) = (\mathrm{B}y_1\,dc_1 + \mathrm{C}y_2\,dc_2)(\mathrm{A}x + \mathrm{B}y_1c_1 + \mathrm{C}y_2c_2).$$

Unless

$$\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1 = 0 \quad \text{and} \quad \mathrm{A}x + \mathrm{B}y_1c_1 + \mathrm{C}y_2c_2 = 0,$$

this may be reduced to

$$(\mathrm{B}c_1\,dc_1 + \mathrm{C}c_2\,dc_2)(\mathrm{A}x + \mathrm{B}y_1c_1 + \mathrm{C}y_2c_2) = (\mathrm{B}y_1\,dc_1 + \mathrm{C}y_2\,dc_2)(\mathrm{A} + \mathrm{B}c_1^2 + \mathrm{C}c_2^2).$$

This last equation only contains x, y_1, y_2, in the combinations $y_1 - c_1x$, $y_2 - c_2x$, and the same will be true for the other equation of the same form that may be derived from the second equation of the complete primitive.

From these two equations and those of the complete primitive we can eliminate the ratio $dc_1 : dc_2$, and the expressions $y_1 - c_1x$ and $y_2 - c_2x$, so as to have an equation in c_1 and c_2 only. In accordance with § 14 the first singular solution thus given will be included in the complete primitive, and the only proper first singular solutions are given by taking

$$ax^2 + by_1^2 + cy_2^2 - 1 = 0 \text{ and } ax + by_1c_1 + cy_2c_2 = 0,$$

or

$$\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1 = 0 \text{ and } \mathrm{A}x + \mathrm{B}y_1c_1 + \mathrm{C}y_2c_2 = 0.$$

§ 23. In order to integrate for the first singular solution, it will be useful to transform the equations to others in terms of t, u, v, the values of μ which satisfy the equation

$$\frac{\mathrm{A}ax^2}{a + \mu\mathrm{A}} + \frac{\mathrm{B}by_1^2}{b + \mu\mathrm{B}} + \frac{\mathrm{C}cy_2^2}{c + \mu\mathrm{C}} - \frac{1}{1 + \mu} = 0.$$

The differential equations are then reduced* to

$$\Sigma \frac{(dt)^2}{(u-v)\,t\,(1+t)\,(a+t\mathrm{A})\,(b+t\mathrm{B})\,(c+t\mathrm{C})} = 0,$$

and

$$\Sigma \frac{(dt)^2}{(u-v)\,(1+t)\,(a+t\mathrm{A})\,(b+t\mathrm{B})\,(c+t\mathrm{C})} = 0.$$

If $\mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1 = 0$, then t, u, or v vanishes, say t; the first equation is satisfied, and the second is reduced to

$$\frac{u\,(du)^2}{(1+u)\,(a+u\mathrm{A})\,(b+u\mathrm{B})\,(c+u\mathrm{C})} = \frac{v\,(dv)^2}{(1+v)\,(a+v\mathrm{A})\,(b+v\mathrm{B})\,(c+v\mathrm{C})}.$$

The variables are here separated, so that we have one first singular solution.

The other is given by taking

$$ax^2 + by_1^2 + cy_2^2 - 1 = 0, \text{ that is } t = \infty,$$

and

$$\frac{(du)^2}{u\,(1+u)\,(a+u\mathrm{A})\,(b+u\mathrm{B})\,(c+u\mathrm{C})} = \frac{(dv)^2}{v\,(1+v)\,(a+v\mathrm{A})\,(b+v\mathrm{B})\,(c+v\mathrm{C})}.$$

Either of these two solutions gives as the second singular solution

$$\left.\begin{aligned} ax^2 + by_1^2 + cy_2^2 - 1 &= 0, \\ \mathrm{A}x^2 + \mathrm{B}y_1^2 + \mathrm{C}y_2^2 - 1 &= 0. \end{aligned}\right\}$$

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* In carrying out the reduction we may use the formula

$$\frac{\mathrm{A}ax^2}{a+\mu\mathrm{A}} + \frac{\mathrm{B}by_1^2}{b+\mu\mathrm{B}} + \frac{\mathrm{C}cy_2^2}{c+\mu\mathrm{C}} - \frac{1}{1+\mu} = \frac{(a-\mathrm{A})\,(b-\mathrm{B})\,(c-\mathrm{C})\,(\mu-t)\,(\mu-u)\,(\mu-v)}{(a+\mu\mathrm{A})\,(b+\mu\mathrm{B})\,(c+\mu\mathrm{C})\,(1+\mu)\,(1+t)\,(1+u)\,(1+v)}$$

and differentiate it, considering μ as constant and t, u, v, y_1, y_2 as functions of x.

The expression

$$\frac{\mathrm{A}a}{a+\mu\mathrm{A}}\,(dx)^2 + \frac{\mathrm{B}b}{b+\mu\mathrm{B}}\,(dy_1)^2 + \frac{\mathrm{C}c}{c+\mu\mathrm{C}}\,(dy_2)^2$$

may then be expressed in terms of dt, du, dv by means of the formulæ for x, y_1, y_2, in terms of t, u, v.

It will thus be found that the equation

$$\left(\frac{\mathrm{A}ax}{a+\mu\mathrm{A}} + \frac{\mathrm{B}by_1p_1}{b+\mu\mathrm{B}} + \frac{\mathrm{C}cy_2p_2}{c+\mu\mathrm{C}}\right)^2 = \left(\frac{\mathrm{A}ax^2}{a+\mu\mathrm{A}} + \frac{\mathrm{B}by_1^2}{b+\mu\mathrm{B}} + \frac{\mathrm{C}cy_2^2}{c+\mu\mathrm{C}} - \frac{1}{1+\mu}\right) \times \left(\frac{\mathrm{A}a}{a+\mu\mathrm{A}} + \frac{\mathrm{B}bp_1^2}{b+\mu\mathrm{B}} + \frac{\mathrm{C}cp_2^2}{c+\mu\mathrm{C}}\right)$$

reduces to

$$\Sigma \frac{dt^2}{(u-v)\,(t-\mu)\,(1+t)\,(a+\mathrm{A}t)\,(b+\mathrm{B}t)\,(c+\mathrm{C}t)} = 0.$$

§ 24. In order to find the other second singular solution let us write T, U, V for $(u-v)(1+t)(a+t\mathrm{A})(b+t\mathrm{B})(c+t\mathrm{C})$, and the two symmetrical expressions.

The differential equations, cleared of fractions, may then be written

$$\mathrm{UV}uv+\mathrm{VT}vt\left(\frac{du}{dt}\right)^2+\mathrm{TU}tu\left(\frac{dv}{dt}\right)^2=0,$$

$$\mathrm{UV}+\mathrm{VT}\left(\frac{du}{dt}\right)^2+\mathrm{TU}\left(\frac{dv}{dt}\right)^2=0.$$

The Jacobian of these expressions with respect to du/dt and dv/dt is

$$4\,du/dt\,dv/dt\,\mathrm{T}^2\mathrm{UV}t\,(v-u),$$

the square of which reduces to a constant multiple of

$$\mathrm{T}^2\mathrm{U}^3\mathrm{V}^3uv\,(t-u)\,(t-v).$$

The factors of this expression are to be considered in turn.

Now the vanishing of such a factor as $1+t$ or $a+\mathrm{A}t$ only causes two solutions of the equations giving x, y_1, y_2 in terms of t, u, v to coincide and only yields a solution of the transformed, not of the original, equations.

The solutions given by supposing t, u or v to vanish or be infinite have been considered. The case when two of the three are equal is left.

If $t=u$, then $\mathrm{V}=0$, so that the equations give

$$\mathrm{TU}\,(dv/dt)^2=0.$$

Suppose first that v is constant.

It is easily found, as in the theory of confocal conicoids, that

$$x^2=\frac{(b-\mathrm{B})(c-\mathrm{C})}{\mathrm{A}a\,(a\mathrm{B}-\mathrm{A}b)\,(a\mathrm{C}-\mathrm{A}c)}\cdot\frac{(a+\mathrm{A}t)\,(a+\mathrm{A}u)\,(a+\mathrm{A}v)}{(1+t)\,(1+u)\,(1+v)}$$

with like values for $y_1{}^2$ and $y_2{}^2$.

Thus if $t=u$ and v is constant, x, y_1, y_2 are constant multiples of

$$\frac{a+\mathrm{A}t}{1+t},\ \frac{b+\mathrm{B}t}{1+t},\ \frac{c+\mathrm{C}t}{1+t}.$$

Hence y_1 and y_2 are linear functions of x and p_1, p_2 are constants connected by a single relation since they involve the arbitrary constant v. This solution is therefore included in the complete primitive and must be the same as was rejected in § 22.

§ 25. If $t = u$ and v is not constant, then $\mathrm{TU} = 0$ and $v = t$, or else $(1 + t)(a + \mathrm{A}t)(b + \mathrm{B}t)(c + \mathrm{C}t) = 0$.

The latter condition leads to no solution.

For the second singular solution we must take $t = u = v$, whence we find that each determinant of the matrix

$$\begin{vmatrix} \alpha x^{\frac{2}{3}}, & \beta y_1^{\frac{2}{3}}, & \gamma y_2^{\frac{2}{3}}, & 1 \\ a, & b, & c, & 1 \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & 1 \end{vmatrix} = 0,$$

if α, β, γ are the cube roots of

$$\frac{\mathrm{A}a(a\mathrm{B} - \mathrm{A}b)(a\mathrm{C} - \mathrm{A}c)}{(b - \mathrm{B})(c - \mathrm{C})}, \text{ \&c.}$$

§ 26. In the geometrical interpretation it will fix the ideas if we suppose the two conicoids projected into confocals.

The complete primitive represents their common tangents. The first singular solution gives the geodesics on each that touch the other, and the second includes the curve of intersection, which is the envelope of these geodesics.

The common tangent planes envelope a torse, whose generators will be common tangent lines. Of the four common tangents to the surfaces from any point of the torse, two will coincide with the generator of the torse. The equation to this torse is $t = u$, and every generator of it is a generator of one confocal of the system. Thus the equations $t = u$, $v =$ const. represent the different generators of this torse, and their occurrence as an apparent singular solution is accounted for.

The cuspidal edge of this torse is a second singular solution, and is represented by the equations

$$\begin{vmatrix} \alpha x^{\frac{2}{3}}, & \beta y_1^{\frac{2}{3}}, & \gamma y_2^{\frac{2}{3}}, & 1 \\ a, & b, & c, & 1 \\ \mathrm{A}, & \mathrm{B}, & \mathrm{C}, & 1 \end{vmatrix} = 0.$$

The General Case. (§§ 27–38.)

§ 27. The straight lines

$$y_1 = c_1 x + b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$y_2 = c_2 x + b_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where the quantities b_1, b_2, c_1, c_2 are connected by two relations, but are otherwise arbitrary, form a congruency, and, if the relations are properly chosen, may be made to represent any congruency.

Thus the lines of any congruency satisfy differential equations of CLAIRAUT'S form.

§ 28. If two consecutive lines meet, the equations (1) and (2), with

$$0 = x dc_1 + db_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$0 = x dc_2 + db_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

must form a consistent system.

Hence

$$db_1 dc_2 - db_2 dc_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

This equation gives two values for the ratio $dc_1 : dc_2$, and shows, therefore, that each line of the system meets two consecutive lines (SALMON, "Geometry of Three Dimensions," § 456).

We are supposing that b_1, b_2 are regarded as functions of c_1, c_2.

§ 29. The elimination of c_1, c_2, dc_1, dc_2 from the equations (1), (2), (3), (4) will give the equation to a surface, and the tangent plane to this surface will contain the line (1), (2). For the equation to the tangent plane is found by eliminating dc_1, dc_2, $d\lambda$ from the differentials of (1) and (2) and of

$$x + \frac{\partial b_1}{\partial c_1} + \lambda \frac{\partial b_2}{\partial c_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\frac{\partial b_1}{\partial c_2} + \lambda \left(x + \frac{\partial b_2}{\partial c_2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

and substituting $\text{X} - x$, $\text{Y}_1 - y_1$, $\text{Y}_2 - y_2$ for dx, dy_1, dy_2. (6) and (7) are found from (3) and (4) by the use of the undetermined multiplier λ.

From (1) and (2) we have, considering c_1 and c_2 and therefore also b_1 and b_2 as functions of x, y_1, y_2,

$$dy_1 = c_1\, dx + x\, dc_1 + db_1,$$
$$dy_2 = c_2\, dx + x\, dc_2 + db_2,$$

and therefore in virtue of (6) and (7)

$$dy_1 + \lambda\, dy_2 = (c_1 + \lambda c_2)\, dx.$$

The equation to the tangent plane is therefore

$$\begin{aligned} \text{Y}_1 - c_1\text{X} + \lambda(\text{Y}_2 - c_2\text{X}) &= y_1 - c_1 x + \lambda(y_2 - c_2 x) \\ &= b_1 + \lambda b_2, \end{aligned}$$

and the tangent plane contains the line (1), (2).

§ 30. Hence if the equation (5) gives unequal values for $dc_1 : dc_2$, the lines of the congruency are bitangents to the surface whose equation is found by eliminating c_1, c_2, λ from (1), (2), (6), (7).

The singular solutions of the differential equations are the solutions of (5), that is, of

$$\frac{\partial b_1}{\partial c_2} dc_2^2 + \left(\frac{\partial b_1}{\partial c_1} - \frac{\partial b_2}{\partial c_2}\right) dc_1\, dc_2 - \frac{\partial b_2}{\partial c_1} dc_1^2 = 0,$$

an ordinary differential equation connecting c_1 and c_2, from the solution of which two equations are to be found free from c_1 and c_2 by help of (1), (2), (6), and (7).

§ 31. The equation connecting λ, c_1, c_2 is

$$\lambda^2 \frac{\partial b_2}{\partial c_1} + \lambda \left(\frac{\partial b_1}{\partial c_1} - \frac{\partial b_2}{\partial c_2}\right) - \frac{\partial b_1}{\partial c_2} = 0.$$

Now

$$\begin{aligned} dy_1 &= c_1\, dx + x\, dc_1 + db_1 \\ &= c_1\, dx - \lambda \frac{\partial b_2}{\partial c_1} dc_1 + \frac{\partial b_1}{\partial c_2} dc_2, \text{ by (6)}, \\ dy_2 &= c_2\, dx + \frac{\partial b_2}{\partial c_1} dc_1 - \frac{1}{\lambda} \frac{\partial b_1}{\partial c_2} dc_2, \text{ by (7)}. \end{aligned}$$

But if the differential equation is to be satisfied we must have

$$dy_1 = c_1\, dx,\ dy_2 = c_2\, dx,$$

and therefore

$$dc_1 + \mu\, dc_2 = 0,$$

where

$$\lambda\mu \frac{\partial b_2}{\partial c_1} = -\frac{\partial b_1}{\partial c_2},$$

so that μ is the second root of the quadratic for λ. This can only hold at one point of contact.

§ 32. The integral of the equation (5) thus represents a double system of curves on the surface, one curve being traced by each point of contact of the double tangent. One curve is such that every tangent to it touches the surface again. This is the first singular solution. The other curve is the locus of the other points of contact of the tangents to the first and is not a solution.* One curve of each system goes

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* In the case of § 17, this curve is found to be the straight line

$$\begin{aligned} y_1 &= x(\mu^2 - 2\mu) + 3\mu^2 - 2\mu^3, \\ y_2 &= -x\mu^2 + 2\mu^3, \end{aligned}$$

and it is included in the complete primitive.

This will always happen if the surface is developable, for the first singular solution is then the section of through each point of the surface for every double tangent that can be drawn to touch the surface there.

§ 33. As to the second singular solution, it is apparently given by the equation $\lambda = \mu$ (these are the two values given for $-dc_1/dc_2$), or

$$\left(\frac{\partial b_1}{\partial c_1} - \frac{\partial b_2}{\partial c_2}\right)^2 + 4\frac{\partial b_1}{\partial c_2} \cdot \frac{\partial b_2}{\partial c_1} = 0.$$

If there is such a solution, which is not generally to be expected, it will represent a curve on the surface every tangent to which meets the surface in four consecutive points. Such tangents to the surface do not, however, generally envelope a curve on it.

Case of a Nodal Curve. (§ 34.)

§ 34. The case of a surface having a nodal curve deserves consideration.

The tangent to such a curve meets the surface in four consecutive points, and is therefore to be counted as a bitangent. The curve satisfies the same differential equations as the bitangents, and the two values of x given by the elimination of the differentials from (3) and (4) are equal, so that the curve is to be reckoned as derived from the singular solution of (5), or as a second singular solution of (1) and (2).

But here a paradox presents itself. There are two tangent planes, and therefore two values of λ, whereas the equations (6) and (7) only yield one unless they are identities.

We have then

$$x = -\frac{\partial b_1}{\partial c_1} = -\frac{\partial b_2}{\partial c_2},$$

$$\frac{\partial b_2}{\partial c_1} = \frac{\partial b_1}{\partial c_2} = 0.$$

Generally, these equations are not consistent, and therefore, generally, there will be no nodal curve.

But on the other hand it will generally be possible to find a singly infinite series of values of x, y_1, y_2, such that the equations (1) and (2), solved for b_1, b_2, c_1, c_2, shall have two pairs of coincident solutions, and the corresponding curve will be a nodal curve on the surface.

The explanation of the paradox is that the surface will generally have other bitangents as well as those belonging to the given congruency. These will form

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the surface by a variable tangent plane and the tangent to this section touches the surface again at a point on the generator along which the plane touches the surface. The generator is therefore the locus of the second point of contact, and as it is the intersection of consecutive tangent planes it is included in the complete primitive.

another congruency, satisfying another pair of differential equations, which the nodal curve also satisfies.

For instance, the normals to an ellipsoid are bitangents to the surface of centres, but there are other bitangents, which form three more congruencies (see SALMON, 'Geometry of Three Dimensions,' § 511*a*). The double curve satisfies the differential equations to the congruency of "synnormals."

If we reciprocate we find the same paradox in relation to the double tangent planes. The line joining the points of contact will generally belong to the second congruency, and the cuspidal edge of the torse which it generates will satisfy the differential equations to this congruency, and belong to its second singular solution, the corresponding first being included in the complete primitive.

Case of a Cuspidal or Parabolic Curve. (§ 35.)

§ 35. At first sight it would appear as if a cuspidal curve ought to satisfy the differential equations to the bitangents, since any tangent to it meets the surface in four consecutive points. But the consideration of the reciprocal surface shows that the tangent planes drawn through such a tangent include three coincident ones, not two distinct coincident pairs. The tangents to a cuspidal curve and the inflexional tangents at parabolic points are therefore not to be counted as bitangents.

When a surface is varied continuously so that a nodal curve changes into a cuspidal, some of the bitangents become chords of the cuspidal curve, and among them are to be reckoned the tangents to that curve. In the same way the inflexional tangents at parabolic points are included in the congruency formed by the intersections of tangent planes at pairs of parabolic points. Thus in a sense the congruency of chords of the cuspidal curve, and that of double tangents to the torse enveloped by the tangent planes at parabolic points, are limiting forms of bitangential congruencies belonging to the surface, though they cannot be considered as true bitangential congruencies belonging to it.

Digression on a Certain Singularity of Surfaces. (§§ 36, 37.)

§ 36. If, however, the curve is both cuspidal and inflexional, in a sense which we shall shortly explain, the tangents to it are true bitangents to the surface.

If we suppose the inflexional tangent at each parabolic point to coincide with the tangent to the parabolic curve, then that curve must be plane or else a double curve on the surface. For the tangent plane at each parabolic point coincides with that at the consecutive point along the inflexional tangent, and hence the tangent plane is the same all along the parabolic curve. A double curve will, however, satisfy the conditions, if we consider the tangent plane at each point as indeterminate. But a nodal curve will be an irrelevant solution, unless it has the singularity in question on one or other of the two sheets. A cuspidal curve may yield us the true solution.

Now it is easily proved that the osculating plane of a curve traced on a surface will coincide with the tangent plane to the surface when it touches one of the inflexional tangents, and only then. The osculating planes of the curve we are discussing will, therefore, be the tangent planes to the surface at the points where they osculate the curve.

Thus the singularity under discussion consists of a cuspidal curve, such that the osculating plane at every point of it touches the surface at that point. The corresponding singularity on the reciprocal surface is of the same kind.

§ 37. To prove this write the equation to the surface (multiplied by a factor, it may be) in the form $\lambda\phi^2 = \mu\psi^3$, λ, μ, ϕ, ψ being functions of x, y, z, such that $\phi = \psi = 0$ are the equations to the cuspidal curve.

Put

$$\phi \div \mu\psi = t,$$

then

$$\psi = t^2\lambda\mu, \qquad \phi = t^3\lambda\mu^2.$$

Suppose now that

$$\phi = z + \phi_2 + \phi_3 + \ldots$$

$$\psi = y + \psi_2 + \psi_3 + \ldots$$

ϕ_r and ψ_r being homogeneous in z, y, x and of the degree r.

Then we may deduce expansions for y and z in ascending powers of t and x as follows—

$$\begin{aligned} y = {} & \alpha t^2 + \beta t^3 + \ldots \\ & + x(\gamma t^2 + \ldots) \\ & + x^2(\delta + \epsilon t^2 + \ldots) \ldots \end{aligned}$$

$$\begin{aligned} z = {} & \alpha' t^3 + \beta' t^4 + \ldots \\ & + x(\gamma' t^2 + \ldots) \\ & + x^2(\delta' + \epsilon' t^2 + \ldots) + x^3(\zeta' + \ldots), \end{aligned}$$

the first power of t being absent throughout.

More generally, we have in the neighbourhood of a cuspidal curve ($t = 0$) expansions of the form

$$x = x_0 + tx_1 + t^2x_2 + \ldots$$

$$y = y_0 + ty_1 + t^2y_2 + \ldots$$

$$z = z_0 + tz_1 + t^2z_2 + \ldots$$

$$w = w_0 + tw_1 + t^2w_2 + \ldots$$

where $x_0, y_0 \ldots x_1, y_1 \ldots$ are functions of a second variable u, such that all the determinants of the matrix

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_0, & y_0, & z_0, & w_0 \\ x_0', & y_0', & z_0', & w_0' \end{vmatrix}$$

vanish, dashes being used to indicate differentiation with respect to u, and dots differentiation with respect to t.

If ξ, η, ζ, ω are the determinants of the matrix

$$\begin{vmatrix} x, & y, & z, & w \\ \dot{x}, & \dot{y}, & \dot{z}, & \dot{w} \\ x', & y', & z', & w' \end{vmatrix},$$

then the reciprocal surface is the locus of $(\xi, \eta, \zeta, \omega)$.

We will now find whether the curve $t = 0$ on the reciprocal surface can be cuspidal.

If ξ, η, ζ, ω are expanded in powers of t we have at once $\xi_0 = 0$, $\eta_0 = 0$, $\zeta_0 = 0$, $\omega_0 = 0$, ξ_r denoting the coefficient of t^r in ξ, and so on.

We also find

$$x_0 \xi_1 + y_0 \eta_1 + z_0 \zeta_1 + w_0 \omega_1 = 0,$$
$$x_0 \xi_2 + y_0 \eta_2 + z_0 \zeta_2 + w_0 \omega_2 = 0,$$
$$x_0' \xi_1 + y_0' \eta_1 + z_0' \zeta_1 + w_0' \omega_1 = 0,$$
$$x_0' \xi_2 + y_0' \eta_2 + z_0' \zeta_2 + w_0' \omega_2 = 2 \begin{vmatrix} x_0', & y_0', & z_0', & w_0' \\ x_0, & y_0, & z_0, & w_0 \\ x_2, & y_2, & z_2, & w_2 \\ x_1', & y_1', & z_1', & w_1' \end{vmatrix} = 2\Delta_1, \text{ say.}$$

and hence, differentiating the first and using the third,

$$x_0 \xi_1' + y_0 \eta_1' + z_0 \zeta_1' + w_0 \omega_1' = 0.$$

Also

$$x_0'' \xi_1 + y_0'' \eta_1 + z_0'' \zeta_1 + w_0'' \omega_1 = 2 \begin{vmatrix} x_0'', & y_0'', & z_0'', & w_0'', \\ x_0, & y_0, & z_0, & w_0, \\ x_2, & y_2, & z_2, & w_2, \\ x_0', & y_0', & z_0', & w_0', \end{vmatrix} = -2\Delta, \text{ say,}$$

so that

$$x_0' \xi_1' + y_0' \eta_1' + z_0' \zeta_1' + w_0' \omega_1' = 2\Delta.$$

If λ, μ are quantities such that

$$x_1 = \lambda x_0 + \mu x_0', \ y_1 = \lambda y_0 + \mu y_0', \text{ etc.},$$

then it is clear that $\Delta_1 = \mu\Delta$.

Hence, if $\Delta = 0$, all the determinants of the matrix

$$\begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_1', & \eta_1', & \zeta_1', & \omega_1' \end{vmatrix} \text{ vanish.}$$

This is the condition that the curve $t = 0$ should be cuspidal on the reciprocal surface, for the common factor t has to be taken out of the expressions for ξ, η, ζ, ω, and the expressions for the coordinates become

$$\xi / t = \xi_1 + \xi_2 t + \ldots \text{ etc.}$$

We find, moreover, that

$$x_0\xi_3 + y_0\eta_3 + z_0\zeta_3 + w_0\omega_3 = - \mu^2\Delta = 0,$$

so that the reciprocal surface satisfies the condition

$$\begin{vmatrix} \xi_3, & \eta_3, & \zeta_3, & \omega_3 \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_1', & \eta_1', & \zeta_1', & \omega_1' \\ \xi_1'', & \eta_1'', & \zeta_1'', & \omega_1'' \end{vmatrix} = 0,$$

which is of the same form as $\Delta = 0$.

Hence this singularity is of the same kind as its reciprocal. The tangents to a curve of this kind are true bitangents to the surface, since they meet it in four consecutive points, and their reciprocals meet the reciprocal surface in four consecutive points.

Lastly the condition $\Delta = 0$ is that which must be satisfied if the tangent plane to the surface, namely

$$\begin{vmatrix} x, & y, & z, & w \\ x_0, & y_0, & z_0, & w_0 \\ x_0', & y_0', & z_0', & w_0' \\ x_2, & y_2, & z_2, & w_2 \end{vmatrix} = 0,$$

coincides with the osculating plane of the cuspidal curve, that is

$$\begin{vmatrix} x, & y, & z, & w \\ x_0, & y_0, & z_0, & w_0 \\ x_0', & y_0', & z_0', & w_0' \\ x_0'', & y_0'', & z_0'', & w_0'' \end{vmatrix} = 0.$$

This proves the theorem.

It is clear that the cuspidal edges of developable surfaces belong to this category, and thus the second singular solution in Example I. is accounted for.

Example III. (§ 38.)

§ 38. As an example, we will take the system of lines represented by the differential equations

$$y_1 = p_1 x + \tfrac{1}{3} p_2^3,$$
$$y_2 = p_2 x + \tfrac{1}{2} p_1^2.$$

It will be more convenient to use y, z, p, q instead of y_1, y_2, p_1, p_2.

The complete primitive is clearly

$$y = \mu x + \tfrac{1}{3}\nu^3,$$
$$z = \nu x + \tfrac{1}{2}\mu^2.$$

The singular solutions are given by the system

$$x\,d\mu + \nu^2\,d\nu = 0,$$
$$x\,d\nu + \mu\,d\mu = 0.$$

We find

$$x = \mu^{\frac{1}{2}}\nu,$$
$$\mu^{\frac{1}{2}}\,d\mu + \nu\,d\nu = 0,$$
$$\tfrac{2}{3}\mu^{\frac{3}{2}} + \tfrac{1}{2}\nu^2 = c, \text{ a constant.}$$

Thus, for the first singular solution,

$$x^3 = \tfrac{3}{4}\nu^3 (2c - \nu^2),$$
$$y = \mu x + \tfrac{1}{3}\nu^3$$
$$= \tfrac{1}{12}\nu (18c - 5\nu^2),$$
$$z = \nu x + \tfrac{1}{2}\mu^2 = \frac{x}{8\nu}(6c + 5\nu^2).$$

By eliminating ν from these equations we have the two equations to a curve belonging to the first singular solution, that is to an envelope of the bitangents.

If now we take the other value of x given by the quadratic $x^2 = \mu\nu^2$ we find for the other point of contact

$$x^3 = -\tfrac{3}{4}\nu^3(2c - \nu^2),$$

$$y = -\tfrac{1}{12}\nu(18c - 13\nu^2),$$

$$z = -\frac{x}{8\nu}(6c - 11\nu^2).$$

The elimination of ν from these equations gives the locus of the other point of contact.

There is no second singular solution.

The equation to the surface which has the lines for bitangents is to be found by eliminating μ, ν from

$$x^2 = \mu\nu^2$$

$$y = \mu x + \tfrac{1}{3}\nu^3$$

$$z = \nu x + \tfrac{1}{2}\mu^2.$$

It is

$$3125x^{12} - 9000x^7yz^2 + 8100x^5y^3z + 2592x^4z^5 - 1944x^3y^5 - 2592x^2y^2z^4 + 648y^4z^3 = 0.$$

Any point of this surface may be represented by the coordinates

$$\left(\xi,\quad \frac{\xi^3}{\nu^2} + \tfrac{1}{3}\nu^3,\quad \nu\xi + \tfrac{1}{2}\frac{\xi^4}{\nu^4}\right).$$

The direction cosines of the tangent plane are proportional to

$$\left(\nu^3 - \frac{\xi^3}{\nu^2},\quad \xi,\ -\nu^2\right).$$

Hence if the line

$$y = kx + l$$

$$z = mx + n$$

is a tangent to the surface at this point, we have

$$\frac{\xi^3}{\nu^2} + \tfrac{1}{3}\nu^3 = k\xi + l,$$

$$\nu\xi + \tfrac{1}{2}\frac{\xi^4}{\nu^4} = m\xi + n,$$

$$\nu^3 - \frac{\xi^3}{\nu^2} = -k\xi + m\nu^2,$$

and therefore

$$\tfrac{4}{3}\nu^3 = m\nu^2 + l,$$

$$\tfrac{3}{2}\frac{\xi^4}{\nu^4} = k\frac{\xi^2}{\nu^2} + n.$$

If the straight line is a bitangent, these equations are also satisfied when other values ξ_1, ν_1 are put in the place of ξ, ν respectively.

Hence ν and ν_1 are either the same root, or different roots of the equation

$$\tfrac{4}{3}\nu^3 = m\nu^2 + l,$$

that is,

$$\nu = \nu_1,$$

or else

$$\nu + \nu_1 - \frac{\nu\nu_1}{\nu + \nu_1} = \tfrac{3}{4}m,$$

and

$$-\frac{\nu^2\nu_1^2}{\nu + \nu_1} = \tfrac{3}{4}l.$$

Also ξ/ν and ξ_1/ν_1 are either the same or are different roots of the equation

$$3(\xi/\nu)^4 - 2k(\xi/\nu)^2 - 2n = 0,$$

in which case

$$\xi/\nu = -\xi_1/\nu_1,$$

or else

$$\xi^2/\nu^2 + \xi_1^2/\nu_1^2 = \tfrac{2}{3}k,$$
$$\xi^2\xi_1^2/\nu^2\nu_1^2 = -\tfrac{2}{3}n.$$

The solution

$$\nu = \nu_1, \ \xi = \xi_1$$

is irrelevant; the solution

$$\nu = \nu_1, \ \xi = -\xi_1$$

gives the original congruency, and

$$\nu = \nu_1, \ \xi^2 + \xi_1^2 = \tfrac{2}{3}k\nu^2$$

gives

$$\xi = \xi_1.$$

A new series of bitangents is given by taking

$$\xi/\nu = -\xi_1/\nu_1, \ \nu \neq \nu_1.$$

We have

$$\nu^2 + \nu\nu_1 + \nu_1^2 = \tfrac{3}{4}m(\nu + \nu_1),$$
$$\nu^2\nu_1^2 = -\tfrac{3}{4}l(\nu + \nu_1),$$
$$\frac{\xi^3}{\nu^3} - k\frac{\xi}{\nu} = \frac{l}{\nu} - \tfrac{1}{3}\nu^2,$$
$$\frac{\xi_1^3}{\nu_1^3} - k\frac{\xi_1}{\nu_1} = \frac{l}{\nu_1} - \tfrac{1}{3}\nu_1^2,$$

whence

$$l\left(\frac{1}{\nu} + \frac{1}{\nu_1}\right) = \tfrac{1}{3}(\nu^2 + \nu_1^2),$$

or

$$-4\nu\nu_1 = \nu^2 + \nu_1^2.$$

Hence we deduce that

$$l = -\tfrac{1}{24}m^3.$$

The elimination of ξ, by help of the equation

$$3(\xi/\nu)^4 - 2k(\xi/\nu)^2 - 2n = 0,$$

gives

$$8192n(k^2 - 2n)^2 - 384km^4(18n - k^2) - 243m^8 = 0.$$

These two equations, connecting k, l, m, n, define the second congruency of bitangents to the surface.

A third is given by combining the equations

$$\begin{aligned}
&\nu^2 + \nu\nu_1 + \nu_1^2 = \tfrac{3}{4}m(\nu + \nu_1),\\
&\nu^2\nu_1^2 = -\tfrac{3}{4}l(\nu + \nu_1),\\
&\xi^2/\nu^2 + \xi_1^2/\nu_1^2 = \tfrac{2}{3}k,\\
&\xi^2\xi_1^2/\nu^2\nu_1^2 = -\tfrac{2}{3}n,\\
&\xi^3/\nu^2 + \tfrac{1}{3}\nu^3 = k\xi + l,\\
&\xi_1^3/\nu_1^2 + \tfrac{1}{3}\nu_1^3 = k\xi_1 + l.
\end{aligned}$$

The result of elimination may be expressed by saying that the expression

$$256u^3 + 9u^2(64lm - m^4) + 18u(21l^2m^2 - lm^5) - 9l^2(m^3 - 3l)^2$$

contains the expression

$$27u^2 - 2u(k^3 - 18kn) - 2n(k^2 - 2n)^2$$

as a factor.

The surface has a nodal curve and a cuspidal curve. These are found by expressing the conditions that the equations

$$\begin{aligned}
y &= x^3/\nu^2 + \tfrac{1}{3}\nu^3\\
z &= \nu x + \tfrac{1}{2}x^4/\nu^4,
\end{aligned}$$

solved for ν, may have a pair of common roots.

If the roots are different we have a nodal curve, to wit, the curve traced by the point

$$(\tfrac{4}{9}t^5,\ \tfrac{40}{81}t^9,\ \tfrac{40}{81}t^8)$$

for different values of t.

If the roots are equal we have the cuspidal curve traced by the point

$$(8t^5, \tfrac{160}{3}t^9, 40t^8).$$

It is easily verified that the tangents to either of these curves are included in the third congruency of bitangents to the surface, and that accordingly the curves will satisfy the differential equations to that congruency.

The cuspidal curve on this surface has the property discussed above (§ 36), and it is for this reason that its tangents are included among the bitangents to the surface.

Example IV.—Inflexional Congruencies. (§§ 39 to 50.)

§ 39. When the two values of $dc_1 : dc_2$ given by the equation $db_1dc_2 - dc_1db_2 = 0$ (§ 28) coincide identically, the lines of the congruency are inflexional tangents to a surface.

For if the direction-cosines of the normal to a surface at the point (x, y_1, y_2) are proportional to l, m, n, the directions of the inflexional tangents are given by the equation

$$dxdl + dy_1dm + dy_2dn = 0.$$

Hence, from the last equation of § 29, we find that the directions of the inflexional tangents in that case are given by

$$d\lambda dy_2 = dx\,(dc_1 + \lambda dc_2 + c_2 d\lambda).$$

But the equation

$$\begin{vmatrix} x + \dfrac{\partial b_1}{\partial c_1}, & \dfrac{\partial b_2}{\partial c_1} \\ \dfrac{\partial b_1}{\partial c_2}, & x + \dfrac{\partial b_2}{\partial c_2} \end{vmatrix} = 0$$

has equal roots, so that

$$x = -\tfrac{1}{2}\left(\frac{\partial b_1}{\partial c_1} + \frac{\partial b_2}{\partial c_2}\right),$$

$$\lambda\frac{\partial b_2}{\partial c_1} = -\tfrac{1}{2}\left(\frac{\partial b_1}{\partial c_1} - \frac{\partial b_2}{\partial c_2}\right),$$

$$xdc_2 + db_2 = \left(x + \frac{\partial b_2}{\partial c_2}\right)dc_2 + \frac{\partial b_2}{\partial c_1}dc_1$$

$$= \frac{\partial b_2}{\partial c_1}(dc_1 + \lambda dc_2).$$

Thus

$$dy_2 - c_2dx = \frac{\partial b_2}{\partial c_1}(dc_1 + \lambda dc_2),$$

and the above equation for the inflexional tangents becomes

$$(dy_2 - c_2 dx)\left(\frac{\partial b_2}{\partial c_1}\,d\lambda - dx\right) = 0.$$

Hence one of them is parallel to the plane $y_2 = c_2 x$.

But it must lie in the tangent plane and pass through the point of contact. It is, therefore, the line

$$y_1 = c_1 x + b_1,$$
$$y_2 = c_2 x + b_2,$$

which was to be proved.

It is remarkable that though from this point of view a congruency of inflexional tangents appears to be a particular kind of bitangential congruency, yet when they are considered from the point of view of the surface, the one is as general as the other, and every surface, whose degree is not 2 or 3, has one of each.

Degenerate Inflexional Congruencies. (§ 40.)

§ 40. An interesting question arises as to whether there is a degenerate form of the inflexional congruency when the surface it envelopes is replaced by a curve. In such a case the lines of the congruency that meet the curve at any one point will form a cone, and the cones belonging to consecutive points of the curve must not meet each other, for if they did they would envelope a surface, and the congruency would be of the bitangential kind. The only kind of conical surface that will meet the case is easily seen to consist of one or more planes touching the curve, and the congruency is made up as follows: Planes are drawn through the tangents to a curve according to some fixed law, and lines are drawn through the points of contact in each plane. The planes will envelope a torse on which the curve will lie, and thus the congruency may be said to consist of all the tangents to a surface at the points of a curve on that surface.

The existence of these two kinds of congruency appears to have been overlooked in the classification given by SALMON ('Geometry of Three Dimensions,' § 453). It might also be desirable to break up the first category given there into two, the bitangents to a surface and the bitangents to a torse, that is, the "lines in two planes" of a curve. The third category would then have to be divided into three, according as one, each, or neither, of the surfaces was developable, and the fourth into two.

Consideration of the General Surface. (§§ 41–49.)

§ 41. Take the surface

$$\phi(x, y_1, y_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The line

$$\left.\begin{aligned} y_1 &= c_1x + b_1 \\ y_2 &= c_2x + b_2 \end{aligned}\right\} \qquad \begin{aligned} &(2), \\ &(3), \end{aligned}$$

will be an inflexional tangent if the equations

$$\mathrm{M} = 0, \quad \partial\mathrm{M}/\partial\mu = 0, \quad \partial^2\mathrm{M}/\partial\mu^2 = 0,$$

are satisfied together, where

$$\mathrm{M} \equiv \phi(\mu,\ c_1\mu + b_1,\ c_2\mu + b^2).$$

By eliminating μ and putting p_1 for c_1, $y_1 - p_1x$ for b_1, &c., we have two equations of CLAIRAUT's form satisfied by the lines of the congruency.

§ 42. For the singular solutions we have

$$xdc_1 + db_1 = 0 \quad \ldots \ldots \quad (4).$$

$$xdc_2 + db_2 = 0 \quad \ldots \ldots \quad (5).$$

$$\frac{\partial\mathrm{M}}{\partial c_1}dc_1 + \frac{\partial\mathrm{M}}{\partial c_2}dc_2 + \frac{\partial\mathrm{M}}{\partial b_1}db_1 + \frac{\partial\mathrm{M}}{\partial b_2}db_2 = 0, \quad \left(\text{because } \frac{\partial\mathrm{M}}{\partial\mu} = 0\right) \quad \ldots \quad (6),$$

$$\frac{\partial^2\mathrm{M}}{\partial\mu\,\partial c_1}dc_1 + \frac{\partial^2\mathrm{M}}{\partial\mu\,\partial c_2}dc_2 + \frac{\partial^2\mathrm{M}}{\partial\mu\,\partial b_1}db_1 + \frac{\partial^2\mathrm{M}}{\partial\mu\,\partial b_2}db_2 = 0, \quad \left(\text{because } \frac{\partial^2\mathrm{M}}{\partial\mu^2} = 0\right) \quad . \quad (7),$$

$$\frac{\partial^3\mathrm{M}}{\partial\mu^2\,\partial c_1}dc_1 + \frac{\partial^3\mathrm{M}}{\partial\mu^2\,\partial c_2}dc_2 + \frac{\partial^3\mathrm{M}}{\partial\mu^2\,\partial b_1}db_1 + \frac{\partial^3\mathrm{M}}{\partial\mu^2\,\partial b_2}db_2 + \frac{\partial^3\mathrm{M}}{\partial\mu^3}d\mu = 0 \quad \ldots \quad (8).$$

But

$$\frac{\partial\mathrm{M}}{\partial c_1} = \mu\frac{\partial\mathrm{M}}{\partial b_1}, \quad \frac{\partial\mathrm{M}}{\partial c_2} = \mu\frac{\partial\mathrm{M}}{\partial b_2}.$$

Hence, by help of (4) and (5), (6) gives

$$(\mu - x)\left\{\frac{\partial\mathrm{M}}{\partial b_1}dc_1 + \frac{\partial\mathrm{M}}{\partial b_2}dc_2\right\} = 0,$$

and (7) gives

$$(\mu - x)\left\{\frac{\partial^2\mathrm{M}}{\partial\mu\,\partial b_1}dc_1 + \frac{\partial^2\mathrm{M}}{\partial\mu\,\partial b_2}dc_2\right\} + \frac{\partial\mathrm{M}}{\partial b_1}dc_1 + \frac{\partial\mathrm{M}}{\partial b_2}dc_2 = 0.$$

Therefore

$$\frac{\partial\mathrm{M}}{\partial b_1}dc_1 + \frac{\partial\mathrm{M}}{\partial b_2}dc_2 = 0,$$

and either

$$\mu = x \quad \text{or} \quad \frac{\partial^2 M}{\partial\mu\,\partial b_1}\,dc_1 + \frac{\partial^2 M}{\partial\mu\,\partial b_2}\,dc_2 = 0.$$

Also (8) becomes

$$(\mu - x)\left\{\frac{\partial^3 M}{\partial\mu^2\,\partial b_1}\,dc_1 + \frac{\partial^3 M}{\partial\mu^2\,\partial b_2}\,dc_2\right\} + 2\left\{\frac{\partial^2 M}{\partial\mu\,\partial b_1}\,dc_1 + \frac{\partial^2 M}{\partial\mu\,\partial b_2}\,dc_2\right\} + \frac{\partial^3 M}{\partial\mu^3}\,d\mu = 0.$$

§ 43. First, let $\mu = x$. Then

$$M = \phi(x, y_1, y_2) = 0,$$

$$\frac{\partial M}{\partial\mu} = \frac{\partial\phi}{\partial x} + c_1\frac{\partial\phi}{\partial y_1} + c_2\frac{\partial\phi}{\partial y_2} = 0,$$

$$\frac{\partial^2 M}{\partial\mu^2} = \frac{\partial^2\phi}{\partial x^2} + 2c_1\frac{\partial^2\phi}{\partial x\,\partial y_1} + 2c_2\frac{\partial^2\phi}{\partial x\,\partial y_2} + c_1^{\,2}\frac{\partial^2\phi}{\partial y_1^{\,2}} + 2c_1c_2\frac{\partial^2\phi}{\partial y_1\,\partial y_2} + c_2^{\,2}\frac{\partial^2\phi}{\partial y_2^{\,2}} = 0,$$

$$\frac{\partial M}{\partial b_1}\frac{dc_1}{dx} + \frac{\partial M}{\partial b_2}\frac{dc_2}{dx} = \frac{\partial\phi}{\partial y_1}\frac{dc_1}{dx} + \frac{\partial\phi}{\partial y_2}\frac{dc_2}{dx} = 0.$$

The last equation and the equation (8) are satisfied in virtue of the first three and the equations $p_1 = c_1$, $p_2 = c_2$.

The integral of these equations represents a series of curves on the surface $\phi = 0$, each tangent to each curve being an inflexional tangent to the surface.

Second Singular Solutions. (§§ 44, 45.)

§ 44. If there is a singular solution of these equations, it is given by supposing two consecutive curves of the series to intersect. At their point of intersection the inflexional tangents will then be in the same direction, and the singular solution therefore appears to be the locus of parabolic points on the surface. We shall, however, find that this curve is not a solution at all in general. The consideration of the reciprocal surface suggests that a cuspidal curve may supply a solution. We begin with the parabolic curve.

We take, as a trial solution,

$$\frac{\partial^2\phi}{\partial x^2} + c_1\frac{\partial^2\phi}{\partial x\,\partial y_1} + c_2\frac{\partial^2\phi}{\partial x\,\partial y_2} = \kappa\frac{\partial\phi}{\partial x},$$

$$\frac{\partial^2\phi}{\partial x\,\partial y_1} + c_1\frac{\partial^2\phi}{\partial y_1^{\,2}} + c_2\frac{\partial^2\phi}{\partial y_1\,\partial y_2} = \kappa\frac{\partial\phi}{\partial y_1},$$

$$\frac{\partial^2\phi}{\partial x\,\partial y_2} + c_1\frac{\partial^2\phi}{\partial y_1\,\partial y_2} + c_2\frac{\partial^2\phi}{\partial y_2^{\,2}} = \kappa\frac{\partial\phi}{\partial y_2}.$$

In order to test whether this is a solution, we differentiate totally, multiply by 1, c_1, c_2, and add, and we have

$$\left(\frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial y_1} + p_2 \frac{\partial}{\partial y_2}\right)' \left(\frac{\partial}{\partial x} + c_1 \frac{\partial}{\partial y_1} + c_2 \frac{\partial}{\partial y_2}\right)'^2 \phi + \frac{dc_1}{dx} \cdot \kappa \frac{\partial \phi}{\partial y_1} + \frac{dc_2}{dx} \cdot \kappa \frac{\partial \phi}{\partial y_2} = 0,$$

where the dashes outside the brackets indicate that in differentiation c_1 and c_2 are to be treated as constants.

In this equation the substitution $p_1 = c_1$, $p_2 = c_2$ will make the coefficient of κ disappear, but the other terms will generally not vanish, and therefore the locus of parabolic points is not generally a solution of the differential equations of the congruency.

It is, in fact, generally speaking, the cusp-locus of the first singular solution.

§ 45. Suppose, now, that the surface has a cuspidal curve. It may be shown that this is a solution.

For at any point (x, y_1, y_2) on the surface, the equations

$$y_1 = c_1 x + b_1, \qquad y_2 = c_2 x + b_2,$$
$$M = 0, \qquad \partial M/\partial \mu = 0, \qquad \partial^2 M/\partial \mu^2 = 0$$

are satisfied by putting

$$\mu = x, \qquad b_1 = y_1 - c_1 x, \qquad b_2 = y_2 - c_2 x$$

if c_1, c_2 are determined by the equations

$$\frac{\partial \phi}{\partial x} + c_1 \frac{\partial \phi}{\partial y_1} + c_2 \frac{\partial \phi}{\partial y_2} = 0,$$

$$\frac{\partial^2 \phi}{\partial x^2} + 2c_1 \frac{\partial^2 \phi}{\partial x \partial y_1} + 2c_2 \frac{\partial^2 \phi}{\partial x \partial y_2} + c_1^2 \frac{\partial^2 \phi}{\partial y_1^2} + 2c_1 c_2 \frac{\partial^2 \phi}{\partial y_1 \partial y_2} + c_2^2 \frac{\partial^2 \phi}{\partial y_2^2} = 0.$$

From the latter may be deduced, by differentiation, on the supposition that $dx = dy_1/c_1 = dy_2/c_2$, which is consistent with $\phi = 0$, that

$$\left(\frac{\partial}{\partial x} + c_1 \frac{\partial}{\partial y_1} + c_2 \frac{\partial}{\partial y_2}\right)'^3 \phi + 2\left(\frac{\partial c_1}{\partial x} + c_1 \frac{\partial c_1}{\partial y_1} + c_2 \frac{\partial c_1}{\partial y_2}\right)\left(\frac{\partial^2 \phi}{\partial x \partial y_1} + c_1 \frac{\partial^2 \phi}{\partial y_1^2} + c_2 \frac{\partial^2 \phi}{\partial y_1 \partial y_2}\right)$$
$$+ 2\left(\frac{\partial c_2}{\partial x} + c_1 \frac{\partial c_2}{\partial y_1} + c_2 \frac{\partial c_2}{\partial y_2}\right)\left(\frac{\partial^2 \phi}{\partial x \partial y_2} + c_1 \frac{\partial^2 \phi}{\partial y_1 \partial y_2} + c_2 \frac{\partial^2 \phi}{\partial y_2^2}\right) = 0 \quad . \quad . \quad . \quad (9).$$

The equation (9) holds all over the surface.

Now, at a singular point where there are two coincident tangent planes,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial y_2} = 0,$$

and

$$\left(\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial y_1} + c_2\frac{\partial}{\partial y_2}\right)^2 \phi$$

is a perfect square, so that the first equation for c_1 and c_2 is nugatory, and the second may be written in any of the forms

$$\frac{\partial^2\phi}{\partial x^2} + c_1\frac{\partial^2\phi}{\partial x\,\partial y_1} + c_2\frac{\partial^2\phi}{\partial x\,\partial y_2} = 0,$$

$$\frac{\partial^2\phi}{\partial x\,\partial y_1} + c_1\frac{\partial^2\phi}{\partial y_1^2} + c_2\frac{\partial^2\phi}{\partial y_1\,\partial y_2} = 0,$$

$$\frac{\partial^2\phi}{\partial x\,\partial y_2} + c_1\frac{\partial^2\phi}{\partial y_1\,\partial y_2} + c_2\frac{\partial^2\phi}{\partial y_2^2} = 0.$$

The equation (9) will therefore not contain the differential coefficients of c_1 and c_2, but will be available to determine c_1 and c_2 themselves.

If there is a cuspidal edge these things hold at every point of it. Let m_1, m_2 be the values of p_1, p_2 taken along the edge, and let us write D for the operator

$$\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y_1} + m_2\frac{\partial}{\partial y_2}.$$

Also we may put

$$\frac{\partial^2\phi}{\partial x^2} = \lambda^2,\ \frac{\partial^2\phi}{\partial x\partial y_1} = \lambda\mu_1,\ \frac{\partial^2\phi}{\partial x\partial y_2} = \lambda\mu_2,\ \frac{\partial^2\phi}{\partial y_1^2} = \mu_1^2,\ \frac{\partial^2\phi}{\partial y_1\partial y_2} = \mu_1\mu_2,\ \frac{\partial^2\phi}{\partial y_2^2} = \mu_2^2.$$

The equations giving the inflexional tangents are then

$$\lambda + \mu_1 p_1 + \mu_2 p_2 = 0.$$

$$\frac{\partial^3\phi}{\partial x^3} + 3p_1\frac{\partial^3\phi}{\partial x^2\partial y_1} + 3p_2\frac{\partial^3\phi}{\partial x^2\partial y_2} + 3p_1^2\frac{\partial^3\phi}{\partial x\partial y_1^2} + 6p_1p_2\frac{\partial^3\phi}{\partial x\partial y_1\partial y_2} + 3p_2^2\frac{\partial^3\phi}{\partial x\partial y_2^2}$$
$$+ p_1^3\frac{\partial^3\phi}{\partial y_1^3} + 3p_1^2p_2\frac{\partial^3\phi}{\partial y_1^2\partial y_2} + 3p_1p_2^2\frac{\partial^3\phi}{\partial y_1\partial y_2^2} + p_2^3\frac{\partial^3\phi}{\partial y_2^3} = 0.$$

We shall show that these two equations will, if p_1, p_2 are considered as coordinates, represent a plane cubic and one of its inflexional tangents.

We have

$$\mathrm{D}\frac{\partial^2\phi}{\partial x^2} = 2\lambda\,\mathrm{D}\lambda,\ \&c.$$

Thus,

$$\mathrm{D}\frac{\partial^2\phi}{\partial x^2} + 2m_1\,\mathrm{D}\frac{\partial^2\phi}{\partial x\,\partial y_1} + 2m_2\,\mathrm{D}\frac{\partial^2\phi}{\partial x\,\partial y_2} + m_1^2\,\mathrm{D}\frac{\partial^2\phi}{\partial y_1^2} + 2m_1m_2\mathrm{D}\frac{\partial^2\phi}{\partial y_1\,\partial y_2} + m_2^2\,\mathrm{D}\frac{\partial^2\phi}{\partial y_2^2}$$
$$= 2\,(\lambda + \mu_1 m_1 + \mu_2 m_2)\,(\mathrm{D}\lambda + m_1\,\mathrm{D}\mu_1 + m_2\,\mathrm{D}\mu_2)$$
$$= 0,$$

for the first factor clearly vanishes, since λ, μ_1, μ_2 are proportional to the direction cosines of the tangent plane.

Also

$$\begin{aligned}\mathrm{D}\frac{\partial^2\phi}{\partial x^2} &+ (m_1+p_1)\,\mathrm{D}\frac{\partial^2\phi}{\partial x\,\partial y_1} + (m_2+p_2)\,\mathrm{D}\frac{\partial^2\phi}{\partial x\,\partial y_2} + m_1p_1\,\mathrm{D}\frac{\partial^2\phi}{\partial y_1{}^2}\\ &+ (m_1p_2+m_2p_1)\,\mathrm{D}\frac{\partial^2\phi}{\partial y_1\,\partial y_2} + m_2p_2\,\mathrm{D}\frac{\partial^2\phi}{\partial y_2{}^2}\\ &= (\lambda+\mu_1m_1+\mu_2m_2)(\mathrm{D}\lambda+p_1\,\mathrm{D}\mu_1+p_2\,\mathrm{D}\mu_2)\\ &\quad+(\lambda+\mu_1p_1+\mu_2p_2)(\mathrm{D}\lambda+m_1\,\mathrm{D}\mu_1+m_2\,\mathrm{D}\mu_2)\\ &= (\lambda+\mu_1p_1+\mu_2p_2)(\mathrm{D}\lambda+m_1\,\mathrm{D}\mu_1+m_2\,\mathrm{D}\mu_2),\end{aligned}$$

and

$$\begin{aligned}\mathrm{D}\frac{\partial^2\phi}{\partial x^2} + 2p_1\,\mathrm{D}\frac{\partial^2\phi}{\partial x\,\partial y_1} + 2p_2\,\mathrm{D}\frac{\partial^2\phi}{\partial x\,\partial y_2} + p_1{}^2\,\mathrm{D}\frac{\partial^2\phi}{\partial y_1{}^2} + 2\,p_1p_2\,\mathrm{D}\frac{\partial^2\phi}{\partial y_1\,\partial y_2} + p_2{}^2\,\mathrm{D}\frac{\partial^2\phi}{\partial y_2{}^2}\\ = (\lambda+\mu_1p_1+\mu_2p_2)(\mathrm{D}\lambda+p_1\,\mathrm{D}\mu_1+p_2\,\mathrm{D}\mu_2).\end{aligned}$$

Thus the point (m_1, m_2) lies on the cubic, its polar line is $\lambda+\mu_1p_1+\mu_2p_2=0$, and its polar conic consists of this line and another. Hence the three solutions of the two equations coincide, and we have $p_1=m_1$, $p_2=m_2$.

Thus the cuspidal edge is enveloped by the inflexional tangents, and is a solution of the differential equation of the congruency.

If the surface has a nodal curve the equations $\mathrm{M}=0$, $\partial\mathrm{M}/\partial\mu=0$, $\partial^2\mathrm{M}/\partial\mu^2=0$ are apparently satisfied along it, but these equations, as they stand, are not enough to determine c_1 and c_2, and when c_1 and c_2 are evaluated by means of another differentiation they are not generally equal to p_1 and p_2 taken along the nodal curve. In fact there are two inflexional tangents in each sheet at every point, and the tangent to the nodal curve is not generally the same as any of the four. Hence the nodal curve, as such, is not a solution.*

Another Second Singular Solution. (§ 46–49.)

§ 46. Let us now take the alternative of § 42 and suppose that

$$\frac{\partial^2\mathrm{M}}{\partial\mu\,\partial b_1}\,dc_1+\frac{\partial^2\mathrm{M}}{\partial\mu\,\partial b_2}\,dc_2=0.$$

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* The lines that touch the surface at points on the nodal curve form such a degenerate inflexional congruency as was discussed above (§ 40), and they will satisfy the differential equations to the inflexional congruency of the surface in the unreduced form in which we have used them. The tangents to the nodal curve form a first singular solution included in the complete primitive, and the nodal curve belongs to the second singular solution which also includes the cuspidal edge of the torse enveloped by the tangent planes.

Then as c_1 and c_2 are not both constant, we have $N = 0$ where

$$N = \frac{\partial^2 M}{\partial\mu\,\partial b_1}\frac{\partial M}{\partial b_2} - \frac{\partial^2 M}{\partial\mu\,\partial b_2}\frac{\partial M}{\partial b_1}.$$

This is a further condition connecting b_1, b_2, c_1, c_2, μ. From it we deduce

$$\frac{\partial N}{\partial c_1}dc_1 + \frac{\partial N}{\partial c_2}dc_2 + \frac{\partial N}{\partial b_1}db_1 + \frac{\partial N}{\partial b_2}db_2 + \frac{\partial N}{\partial\mu}d\mu = 0.$$

From this and the former equations we can again eliminate the differentials. Suppose the resultant equation to be $P = 0$. Then the equations $N = 0$, $P = 0$ afford an integral of the differential equations. We will verify this.

§ 47. $N = 0$ may be replaced by

$$\frac{\partial M}{\partial b_1}C_1 + \frac{\partial M}{\partial b_2}C_2 = 0,$$

$$\frac{\partial^2 M}{\partial\mu\,\partial b_1}C_1 + \frac{\partial^2 M}{\partial\mu\,\partial b_2}C_2 = 0,$$

where C_1 and C_2 do not both vanish.

Also $P = 0$ may be replaced by

$$\left(\frac{\partial N}{\partial c_1} - x\frac{\partial N}{\partial b_1}\right)C_1 + \left(\frac{\partial N}{\partial c_2} - x\frac{\partial N}{\partial b_2}\right)C_2 + \frac{\partial N}{\partial\mu}K = 0 \quad . \quad . \quad . \quad . \quad (10),$$

$$(\mu - x)\frac{\partial^3 M}{\partial^2\mu\,\partial b_1}C_1 + \left(\mu - x\right)\frac{\partial^3 M}{\partial^2\mu\,\partial b_2}C_2 + \frac{\partial^3 M}{\partial\mu^3}K = 0 \quad . \quad . \quad . \quad (11).$$

We have also $M = 0$, $\partial M/\partial\mu = 0$, $\partial^2 M/\partial\mu^2 = 0$.

The last three equations give

$$\frac{\partial M}{\partial b_1}\left(\mu c'_1 + b'_1\right) + \frac{\partial M}{\partial b_2}\left(\mu c'_2 + b'_2\right) = 0,$$

if we write c'_1 for dc_1/dx, &c.

We may then put

$$\mu c'_1 + b'_1 = uC_1,\ \mu c'_2 + b'_2 = uC_2.$$

They also give

$$\frac{\partial^2 M}{\partial\mu\,\partial b_1}uC_1 + \frac{\partial^2 M}{\partial\mu\,\partial b_2}uC_2 + c'_1\frac{\partial M}{\partial b_1} + c'_2\frac{\partial M}{\partial b_2} = 0,$$

so that we may put

$$c'_1 = vC_1,\ c'_2 = vC_2,$$
$$b'_1 = (u - \mu v)\, C_1,\ b'_2 = (u - \mu v)\, C_2.$$

Further, they give

$$\frac{\partial^3 M}{\partial\mu^2\,\partial b_1} uC_1 + \frac{\partial^3 M}{\partial\mu^2\,\partial b_2} uC_2 + 2\frac{\partial^2 M}{\partial\mu\,\partial b_1} vC_1 + 2\frac{\partial^2 M}{\partial\mu\,\partial b_2} vC_2 + \frac{\partial^3 M}{\partial\mu^3}\mu' = 0,$$

whence, by comparison with (11),

$$uK = (\mu - x)\,\mu'.$$

Again, since $N = 0$,

$$\frac{\partial N}{\partial c_1} vC_1 + \frac{\partial N}{\partial c_2} vC_2 + \frac{\partial N}{\partial b_1}(u - \mu v)\, C_1 + \frac{\partial N}{\partial b_2}(u - \mu v)\, C_2 + \frac{\partial N}{\partial\mu}\frac{uK}{\mu - x} = 0.$$

Multiply this by $\mu - x$ and (10) by u and subtract.

The equation connecting u and v is then found to be

$$\{u - v(\mu - x)\}\left(\frac{\partial N}{\partial c_1} C_1 + \frac{\partial N}{\partial c_2} C_2 - \mu\frac{\partial N}{\partial b_1} C_1 - \mu\frac{\partial N}{\partial b_2} C_2\right) = 0.$$

Thus in general $u - \mu v = -vx$, that is, $c'_1 x + b'_1 = 0 = c'_2 x + b'_2$ and the differential equations are satisfied.

§ 48. In the case when $N = 0$, $P \neq 0$, we still have

$$dc_1 : dc_2 :: C_1 : C_2 :: db_1 : db_2,$$

but the ratio $dc_1 : db_1$ is unassigned.

The equation to the tangent plane to the surface at

$$(\mu,\ c_1\mu + b_1,\ c_2\mu + b_2)$$

is, as always,

$$(y_1 - c_1 x - b_1)\frac{\partial M}{\partial b_1} + (y_2 - c_2 x - b_2)\frac{\partial M}{\partial b_2} = 0.$$

Hence, when $N = 0$, the tangent plane at any adjacent point is

$$(y_1 - c_1 x - b_1)\left(\frac{\partial M_1}{\partial b_1} + d\frac{\partial M_1}{\partial b_1}\right) + (y_2 - c_2 x - b_2)\left(\frac{\partial M}{\partial b_2} + d\frac{\partial M}{\partial b_2}\right) = 0.$$

Thus the tangent planes at all adjacent points pass through the same straight line

$$\left.\begin{aligned} y_1 &= c_1 x + b_1 \\ y_2 &= c_2 x + b_2 \end{aligned}\right\}.$$

Hence the point is a parabolic point on the surface, and the line of the congruency is the intersection of tangent planes at two consecutive parabolic points. All such lines will generate a torse, and they would belong to the first singular solution were they not already in the complete primitive.

§ 49. The intersection of two consecutive generators of this torse is given by the equations

$$\begin{aligned} y_1 &= c_1x + b_1, \\ y_2 &= c_2x + b_2, \\ x &= -\,db_1/dc_1 \text{ or } -\,db_2/dc_2, \end{aligned}$$

where b_1, c_1, b_2, c_2 are connected by the equations

$$M = 0,\ \partial M/\partial\mu = 0,\ \partial^2M/\partial\mu^2 = 0,\ N = 0.$$

These equations are satisfied if $P = 0$, and therefore the cuspidal edge of this torse is the second singular solution given by taking $N = 0 = P$.

A Particular Example. (§ 50.)

§ 50. As an example of an inflexional congruency, we may take the system of lines

$$\begin{aligned} y &= 3a^2b^4x + \tfrac{1}{3}\,b^9(1 - 6a^3), \\ z &= b^3(1 + 2a^3)\,x - \tfrac{3}{2}\,a^4b^8. \end{aligned}$$

These are half the system of inflexional tangents of the surface of § 38.

It is easily verified that the cuspidal curve is a second singular solution, and the nodal curve not.

The first singular solution is given by

$$x = ab^5, \quad b = \text{constant}.$$

It is

$$y = \frac{x^3}{b^6} + \tfrac{1}{3}\,b^9,$$

$$z = \tfrac{1}{2}\,\frac{x^4}{b^{12}} + b^3x\,.$$

The system

$$y = (1 + a^3)\,b^4\,\frac{x}{a} - \tfrac{2}{3}\,b^9,$$

$$z = 2b^3x + \tfrac{1}{2}\,a^4b^8 - ab^8,$$

which includes the other inflexional tangents to the same surface, would serve equally well.

This example shows that it is possible for the inflexional tangents to a surface to form two distinct congruencies. The parabolic and cuspidal curves, moreover, coincide.

Example V.—System of Curves in Space. (§§ 51, 52.)

§ 51. As another example, take the equations

$$\frac{dx^2}{x^2-1}=\frac{dy^2}{y^2-1}=\frac{dz^2}{z^2-1}.$$

The complete primitive is the result of eliminating t from

$$2x=t+\frac{1}{t}, \quad 2y=at+\frac{1}{at}, \quad 2z=bt+\frac{1}{bt},$$

a and b being the constants of integration.

The curves represented are conics touching the six planes

$$x=\pm 1, \quad y=\pm 1, \quad z=\pm 1.$$

The first singular solution includes six forms—

$$\begin{aligned} x&=\pm 1, & 2y&=u+\frac{1}{u}, & 2z&=cu+\frac{1}{cu};\\ y&=\pm 1, & 2z&=u+\frac{1}{u}, & 2x&=cu+\frac{1}{cu};\\ z&=\pm 1, & 2x&=u+\frac{1}{u}, & 2y&=cu+\frac{1}{cu};\end{aligned}$$

in each u is a variable parameter, and c an arbitrary constant. The curves represented are conics inscribed in the six faces of the cube contained by the planes

$$x=\pm 1, \quad y=\pm 1, \quad z=\pm 1.$$

The second singular solution consists of the twelve edges of this cube.

§ 52. If we seek the singular solutions by means of the differential equations, we take

$$p^2(x^2-1)-(y^2-1)=0,$$
$$q^2(x^2-1)-(z^2-1)=0,$$

and form the Jacobian with respect to p and q.

We thus find

$$pq\,(x^2 - 1)^2 = 0,$$

whence

$$x = \pm 1, \text{ or } y = \pm 1, \text{ or } z = \pm 1.$$

Any one of these is found to be a singular first integral, and to reduce the equations to one of the form

$$p^2\,(x^2 - 1) = y^2 - 1,$$

which we have seen how to integrate.

The singular solution of this is again

$$(x^2 - 1)\,(y^2 - 1) = 0.$$

System of Plane Curves. Another extension of CLAIRAUT'S *Form.* (§§ 53–55.)

§ 53. There is an extension of CLAIRAUT'S form to higher orders, with one dependent variable.

Write p_r for $d^r y/dx^r$, so that p_0 will mean y.

Then integration by parts gives (r being $\leqslant n$),

$$\int p_{n+1}\,x^r\,dx = x^r p_n - r x^{r-1} p_{n-1} + r\,(r-1)\,x^{r-2} p_{n-2} \ldots + (-1)^r\, r\,!\, p_{n-r}.$$

Call this expression q_r, and take the equation,

$$\phi\,(q_0, q_1, q_2 \ldots q_n) = 0.$$

This may be solved at once by differentiating; since

$$dq_r/dx = x^r\, p_{n+1},$$

we have

$$p_{n+1} \left\{ \frac{\partial\phi}{\partial q_0} + x\frac{\partial\phi}{\partial q_1} + x^2 \frac{\partial\phi}{\partial q_2} + \ldots + x^n \frac{\partial\phi}{\partial q_n} \right\} = 0.$$

The first factor gives the complete primitive, which consists of the equations,

$$q_0 = a_0, \quad q_1 = a_1, \ldots q_n = a_n,$$

where $a_0, a_1 \ldots a_n$ are constants, connected by the relation

$$\phi\,(a_0, a_1, \ldots a_n) = 0,$$

but otherwise arbitrary.

The value of y may be found as follows—in the equation

$$a_r\, x^{n-r} = p_n x^n - r p_{n-1} x^{n-1} + r\,(r-1)\,p_{n-2} x^{n-2} \ldots$$

take differences with respect to r; thus

$$\Delta^s\, a_r\, x^{n-r} = (-1)^s\, s\,!\; x^{n-s} p_{n-s} + (-1)^{s+1}\, 2.3 \ldots (s+1)\, r\, x^{n-s-1} p_{n-s-1} + \ldots$$

Put now $s = n$, $r = 0$, and we have finally

$$(-1)^n n!\, y = \Delta^n a_0 x^{n-0} = a_n - n\, a_{n-1} x + \frac{n(n-1)}{2!} a_{n-2} x^2 \ldots + (-1)^n a_0 x^n,$$

the constants being connected by the relation

$$\phi(a_0, a_1, \ldots a_n) = 0.$$

§ 54. The factor

$$\frac{\partial\phi}{\partial q_0} + x\frac{\partial\phi}{\partial q_1} + \ldots + x^n \frac{\partial\phi}{\partial q_n},$$

equated to zero, leads to the singular solutions, the first integral being found by elimination of p_n from $\phi = 0$ by means of it.

Thus the solution of $\phi(q_0, q_1 \ldots q_n) = 0$ is exactly on the lines of that of $\psi(xp_1 - y, p_1) = 0$, which is CLAIRAUT's form.

§ 55. The equations to be integrated in finding the singular solutions are

$$\frac{da_1}{da_0} = \frac{da_2}{da_1} = \frac{da_3}{da_2} = \ldots = \frac{da_n}{da_{n-1}} \;(= x),$$
$$\phi(a_0, a_1, \ldots a_n) = 0.$$

For example, when $n = 2$, we have

$$da_1^2 = da_0\, da_2,$$
$$\phi(a_0, a_1, a_2) = 0.$$

If (a_0, a_1, a_2) are taken as Cartesian coordinates, the solution represents curves on the surface $\phi = 0$, the tangents to which are parallel to generators of the cone $a_1^2 = a_0 a_2$, that is to say, meet a certain curve at infinity. The second singular solution is given by forming the envelope of such curves, which does not generally exist, but may in particular cases.

Example VI. (§ 56.)

§ 56. As an example take the equation

$$(2yp_2 - p_1^2)^3 = 4p_2(p_1 - xp_2)^3, \text{ or } (q_0 q_2 - q_1^2)^3 + 4q_0 q_1^3 = 0, \; n \text{ being } 2.$$

The complete primitive is

$$4ac^3 y = 1 + 2c + 4a^2c^3x + 4a^4c^6x^2.$$

The equations giving singular solutions are

$$(1 + 6a^2c^3x) \,.\, da + 6a^3c^2x \,.\, dc = 0,$$
$$\left(x - \frac{1 + 2c}{2a^2c^3}\right) da - \frac{3 + 4c}{2ac^4}\, dc = 0.$$

The result of eliminating x is

$$c^2\,da^2 + 3\,(c\,da + a\,dc)\,\{(1 + 2c)\,c\,da + (3 + 4c)\,a\,dc\} = 0.$$

The complete primitive of this is

$$\alpha^2 - 6ac^2\alpha - 6a^2c^3 = 0,$$

and for the singular solution of it we must take

$$\alpha = 3ac^2 = -2ac.$$

The solutions $a = 0$, $c = 0$ arise by giving α the particular value zero, and the true singular solution is

$$2 + 3c = 0.$$

Hence for the first singular solution

$$\alpha^2 - 6ac^2\alpha - 6a^2c^3 = 0,$$
$$\alpha^3xy - 3\alpha^2x + 3\alpha y - 1 = 0.$$

For the second

$$x = y^2.$$

This is the equation to a parabola.

The curves of the first singular system are hyperbolas, having their asymptotes parallel to the axes of coordinates and having contact of the second order with this parabola.

The complete primitive represents a series of parabolas with axes parallel to the axis of y, and each having contact of the second order with some one of the hyperbolas, and, in fact, with two of them, since α is given in terms of a and c by a quadratic equation.

§ 57. In order to make up further examples we only need to take :—

(1.) A curve A.

(2.) A series of curves A_1, depending on one parameter, each having contact of order n with the curve A.

(3.) A series of curves A_2, involving two parameters, each having contact of order n with some one of the curves A_1, and so on, till we have a series of curves A_n, involving n parameters, each having contact of order n with some one of the curves A_{n-1}.

Then A_n is the complete primitive of a differential equation of order n, A_{n-1} will be a first singular solution, A_{n-r} an r^{th} singular solution.